The Glueball

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Companion note to *Spectral Geometry of Vacuum Entanglement on Q_3 and Q_4*

The glueball is the sourceless counterpart of the confining string. Where the string is the sourced part of the colour flux and carries the fundamental representation factor of the charges standing at its ends, the glueball is the closed part of the same flux on the same face, and carries the same edge Sakharov trace squared weighted by the gluonic representation factor of a contour with no ends.

A glueball carries no external colour charge, having no endpoints at which charges could sit. The condition on the contour is that the net flow vanish at every vertex,

$$(d_0^{\mathsf{T}} b_g)_v = 0 \quad \text{for every } v, \tag{1}$$

which is the closure of the contour, an open contour being sourced at its endpoints. The condition is not imposed on the spectrum from outside but selects within it: since $L_1 = d_0 d_0^{\mathsf{T}}$, for any edge vector b,

$$L_1 b = 0 \iff \langle b, d_0 d_0^{\mathsf{T}} b\rangle = 0 \iff |d_0^{\mathsf{T}} b|^2 = 0 \iff d_0^{\mathsf{T}} b = 0, \tag{2}$$

so the closed contours and the kernel of the edge operator are the same space, of dimension seventeen by (71). Conversely an eigenvector with $\lambda \neq 0$ has $L_1 b = \lambda b \neq 0$ and is therefore sourced, so the edge space splits as

$$\mathbb{R}^{32} = \ker d_0^{\mathsf{T}} \oplus \operatorname{ran} d_0, \tag{3}$$

the kernel carrying every closed contour and the range, the span of the eigenspaces $\lambda = 2,4,6,8$ of (71), carrying every sourced one.

The colour bundle carries no closed contour on the cube, and none on the prism either while τ is switched off, by the argument of (121): each vertex meets exactly one z-edge, so (1) is the single term $\pm b_g(e_v) = 0$ at that vertex, one equation per vertex setting every z-edge coefficient to zero. The only direction the prism adds is τ, so the face that can carry a closed colour contour is the $z\tau$-face, and on $E_z \cup E_\tau$ there are exactly four such contours, one per (x,y) position. On the face at $(x,y) = (0,0)$, with z-edges $(0000)-(0010)$, $(0001)-(0011)$ and τ-edges $(0000)-(0001)$, $(0010)-(0011)$, the two-dimensional $F_{z\tau}$ parity class splits orthogonally into its sourced part, taken by the string in (122), and its closed part, the circulation around the face, unique up to normalization,

$$\begin{aligned} b_g\big((0000)-(0010)\big) &= +1, \quad b_g\big((0010)-(0011)\big) = +1,\\ b_g\big((0001)-(0011)\big) &= -1, \quad b_g\big((0000)-(0001)\big) = -1, \end{aligned} \tag{4}$$

with $|b_g|^2 = 4$, the normalization cancelling in the weights of (110). Its boundary vanishes at all four face vertices, $(d_0^{\mathsf{T}} b_g)_v = 0$, so it carries no source pattern and no external charge. The closed colour contour is zero on the cube and nonzero on the prism, so the glueball is a prism quantity, as the string tension is.

The flux b_g is closed, so by (2) it lies in the kernel of L_1 and is therefore an eigenvector of M_1 at $\lambda = 0$. It does not spread over the edge spectrum. Its projections into the eigenspaces of L_1, from (123), are concentrated in the one eigenspace,

$$\|P_0 b_g\|^2 = 4, \quad \|P_2 b_g\|^2 = 0, \quad \|P_4 b_g\|^2 = 0, \quad \|P_6 b_g\|^2 = 0, \quad \|P_8 b_g\|^2 = 0, \tag{5}$$

summing to $4 = |b_g|^2$, so the normalized weights $\widehat{c}_\lambda^2 = \|P_\lambda b_g\|^2/4$ are

$$\widehat{c}_0^2 = \frac{4}{4} = 1, \quad \widehat{c}_2^2 = 0, \quad \widehat{c}_4^2 = 0, \quad \widehat{c}_6^2 = 0, \quad \widehat{c}_8^2 = 0, \tag{6}$$

summing to 1. From (110) the propagator weight is the sum over the spectrum, and with one surviving normalized weight the single term remains,

$$G_{b_g} = \sum_\lambda \frac{\widehat{c}_\lambda^2}{\lambda+1} = \frac{1}{0+1} + \frac{0}{2+1} + \frac{0}{4+1} + \frac{0}{6+1} + \frac{0}{8+1} = 1. \tag{7}$$

The weight is unity for every element of the kernel, not for the face cycle alone, the projections of (5) being those of any closed contour. The denominator is the edge Sakharov trace (129), the same trace of the same operator that the string tension divides, so the spectral weight is

$$\Pi_{b_g} = \frac{G_{b_g}}{\Sigma_E} = \frac{1}{6368/315} = \frac{315}{6368} = 0.049466. \tag{8}$$

The flux is a vector, so the spectral weight enters squared, $p = 2$.

The flux carries colour. The numerator of (107) is the charge the flux carries: the confining string is a directed flux running from a source, and carries the fundamental charge of (109), one charge with two ends rather than two independent charges. The closed contour runs from nothing to nothing and carries no charge at all, so its numerator is the singlet, $\mathrm{Tr}_{\text{charge}}(\nVdash) = 1$. This is (107) applied unchanged, the numerator being what the flux carries in both cases.

The denominator is the channels the object propagates through, and confinement changes which channels are open. The confining string carries fundamental colour, so it propagates through the octet of (106), the eight gluons, and $\mathrm{Tr}_{\text{channel}}(\nVdash) = 8$. The closed contour is a colour singlet, and a singlet does not propagate through gluons, which confinement removes from the physical spectrum. Its channels are hadronic. For the scalar there are four, $\pi\pi$, $K\bar{K}$, $\eta\eta$ and $\eta\eta'$, the two-pseudoscalar channels of the light nonet [?], so $\mathrm{Tr}_{\text{channel}}(\nVdash) = 4$,

$$\Theta_{b_g} = \frac{\mathrm{Tr}_{\text{charge}}(\nVdash)}{\mathrm{Tr}_{\text{channel}}(\nVdash)} = \frac{1}{4}. \tag{9}$$

Both traces are dimensions of state spaces, as in (107), and sourcedness fixes both. The Higgs field is colourless in charge and in channel, so $\Theta_\psi = 1/1$. The string is sourced, carrying fundamental colour through the eight gluons, so $\Theta_{b_s} = 3/8$. The glueball is sourceless, a singlet propagating through the four hadronic channels, so $\Theta_{b_g} = 1/4$. The one condition fixes both traces, as it fixed the mode in (1) and the propagator weight in (7). The octet is internal to the construction, the colour assigned to E_z in §3, while the four hadronic channels are those of the light quark nonet and are not derived here.

From the master equation (102), with the spectral weight squared and one power of the representation factor,

$$m_{0^{++}} = m_P\, \Theta_{b_g}\, \Pi_{b_g}^2 \exp(-1/\xi^*), \tag{10}$$

and substituting,

$$\Theta_{b_g}\, \Pi_{b_g}^2 = \frac{1}{4}\left(\frac{315}{6368}\right)^2 = \frac{1}{4}\cdot\frac{99225}{40551424} = \frac{99225}{162205696} = 6.11723\times 10^{-4}, \tag{11}$$

$$m_{0^{++}} = 1.2209\times 10^{19}\ \text{GeV}\cdot 6.11723\times 10^{-4}\cdot 2.31952\times 10^{-16} = 1732\ \text{MeV}, \tag{12}$$

in agreement with the quenched lattice scalar glueball $m(0^{++}) = 1730(50)(80)$ MeV determined from the anisotropic Wilson action [?].