Harmonic Entanglement Theory: Gravity and Gauge Structure from Vacuum Entanglement

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May 28, 2026

Abstract

Harmonic Entanglement Theory (HET) develops the general-relativistic metric and the Standard-Model gauge couplings as emergent from vacuum entanglement organized by a multi-scale entanglement renormalization ansatz (MERA) on the three-cube Q_3. Each bond carries the relative entropy D_{KL} as its coordinate, bounded by the Bekenstein–Hawking ceiling $\log\chi = 1/4$, with the spatial dimension $d = 3$. The partition function under the MERA isometry constraint gives the per-site vacuum entanglement $S^* = 2/9$. The Wilson plaquette action of lattice gauge theory is identified with the plaquette-averaged committed entanglement, $\langle\Omega_\square\rangle = \langle D_{\mathrm{KL}}\rangle_\square = S^*(1 + (\log\chi)^2) = 17/72$, fixing the bare unified coupling $g^2(m_P) = 17/72$ at the strong value $g_s^2(m_P)$. The gauge group is assigned by a single combinatorial operator — the Q_3 curl-Laplacian, whose spectrum $\{0^{(7)}, 4^{(3)}, 6^{(2)}\}$ carries $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, and $\mathrm{SU}(3)_c$. The MERA disentanglers, the unitaries that remove short-range entanglement at each layer, are proposed as the origin of the force carriers — gauge fields on the cube edges, the graviton on its faces — and generate a B_3-equivariant correction on the twelve edges that splits the unified coupling into the distinct sector values. The resulting cluster (g_s^2, g_L^2, g_Y^2, $\sin^2\theta_W$, $\alpha_{\mathrm{em}}^{-1}$) agrees with PDG 2024 run up to m_P at the 0.01% level. In the gravitational sector, the metric perturbation follows from the discrete Ryu–Takayanagi area law, and a spin-2 graviton emerges as a Q_3-selected composite of the entangleon (the continuum quantum of D_{KL}); because the graviton is composite and the bond discreteness supplies a physical ultraviolet regulator, the construction is proposed to address the non-renormalisability of perturbative quantum gravity. Newton's constant $G_N = (2/\pi)\,\ell_P^2$ follows from the Sakharov mechanism, and the induced metric recovers the linearised Einstein equation and the Newtonian limit.

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1 Introduction

The proposal that gauge and gravitational structure arise from quantum entanglement has emerged from several converging programs. Van Raamsdonk has argued that spacetime connectivity is built up from entanglement [1], made quantitative for holographic conformal field theories [2]. Swingle has identified the multi-scale entanglement renormalization ansatz (MERA) as a discrete realisation of holographic geometry [3]. Furey [4] and Szangolies [5] have derived the Standard Model gauge group from division-algebraic and Hopf-fibration substrates on few-qubit entanglement. The closest precedent for deriving the Standard Model gauge group from a geometric structure is Connes' noncommutative geometry programme [6, 7], which recovers the gauge cluster from the unitary group of the algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ via the Spectral Action; that algebra is chosen to match the Standard Model, whereas the cell Q_3 here is fixed by independent inputs (Bekenstein–Hawking bond saturation and the dimensionality of space). These constructions share the view that gauge and gravity are not added by hand but recovered as consequences of quantum information.

Harmonic Entanglement Theory sits within this program with a specific substrate: each entanglement bond is a quantum harmonic oscillator whose coordinate is the Umegaki relative entropy D_{KL}, with bond saturation set at the Bekenstein–Hawking ceiling [8, 9] $\log \chi = 1/4$. From these a binary MERA on the three-cube Q_3 is selected, and its structure is shown to yield the Standard-Model gauge cluster and the emergent metric with Newton's constant.

2 Vacuum Entanglement

2.1 The bond coordinate

An elementary bond of the entanglement network is identified with one Planck area of horizon: the smallest unit through which entropy can flow across a codimension-one surface at the Planck scale. The Bekenstein–Hawking area law [8, 9] $S = A/(4\ell_P^2)$ assigns each Planck area the entropy of one quarter of a nat, fixing the maximum entanglement entropy carried by one bond at

$$\log \chi \;=\; \frac{S_{\mathrm{BH}}}{A}\,\ell_P^2 \;=\; \tfrac{1}{4}. \tag{1}$$

The dimensionless value $\log \chi = 1/4$ is therefore the Bekenstein–Hawking area-law coefficient inherited through the bond–area identification, not a free parameter of the construction.

The entanglement network consists of sites — elementary entanglement nodes, each carrying a Hilbert space of dimension χ — connected pairwise by bonds. Each bond (i, j) of the entanglement network carries a scalar coordinate, the relative entanglement entropy (Umegaki relative entropy [10])

$$D_{\mathrm{KL}}(\rho_{ij} \| \rho_{ij}^0) = \mathrm{tr}\big(\rho_{ij} \log \rho_{ij} - \rho_{ij} \log \rho_{ij}^0\big), \tag{2}$$

the reduced bond state ρ_{ij} measured against a reference ρ_{ij}^0. Two reference states enter and are not interchangeable. The structured MERA vacuum ρ_{ij}^0 has non-flat spectrum and per-site von Neumann entropy S^* ($0 < S^* < \log \chi$); it is the reference of the dynamical coordinate, and $D_{\mathrm{KL}}(\rho_{ij} \| \rho_{ij}^0)$ vanishes exactly at the vacuum. The maximally mixed state $\sigma_{ij}^0 = \mathbb{I}_{\chi^2}/\chi^2$ gives $D_{\mathrm{KL}}(\rho \| \sigma^0) = \log \chi - S(\rho)$, the entropy deficit; it is the reference of the static Bekenstein–Hawking normalisation only. The dynamical bond coordinate is $D_{\mathrm{KL}}(\rho_{ij} \| \rho_{ij}^0)$, the relative entropy against the structured vacuum.

Within the Rényi family $D_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{tr}(\rho^\alpha\sigma^{1-\alpha})$, the coordinate (2) is the $\alpha \to 1$ member, recovered as the L'Hôpital limit $\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = \mathrm{tr}[\rho(\log\rho - \log\sigma)]$. It is the unique member of that family satisfying all four of:

1. **Positive semi-definite:** $D_{\mathrm{KL}} \geq 0$, with equality iff $\rho_{ij} = \rho^0_{ij}$ (Klein's inequality).
2. **Thermodynamic:** $\delta D_{\mathrm{KL}} = \delta\langle K\rangle - \delta S$ with $K = -\log\rho^0$ the modular Hamiltonian (quantum first law of entanglement).
3. **Geometric:** $\delta D_{\mathrm{KL}} = \delta A_{\min}/4\ell_P^2$ (the linearised Ryu–Takayanagi first law [2]).
4. **Bounded:** $D_{\mathrm{KL}} \leq \log\chi$, the Bekenstein–Hawking bond capacity.

Condition 2 is the discriminating one: the first-law variation requires $\mathrm{tr}[\rho\,\delta\log\rho] = 0$, i.e. ρ entering linearly under the trace after variation. For $\alpha \neq 1$, $\delta D_\alpha \propto \mathrm{tr}[\rho^{\alpha-1}\sigma^{1-\alpha}\,\delta\rho]$, which carries the operator $\rho^{\alpha-1}\sigma^{1-\alpha}$ rather than the modular Hamiltonian $K = -\log\sigma$; there is no $\alpha \neq 1$ for which δD_α takes the form $\delta\langle K\rangle - \delta S$, so the first law selects $\alpha = 1$. Conditions 3 and 4 reinforce: the Ryu–Takayanagi bridge requires $K = -\log\sigma$, which enters only at $\alpha = 1$, and the Bekenstein–Hawking bound is saturable only at $\alpha = 1$, where $D_1(\rho\|\mathbb{I}/\chi) = \log\chi - S(\rho)$. The natural non-Rényi alternatives — von Neumann entropy, mutual information, trace distance, fidelity and the Bures metric, and the quantum Fisher information — each fail at least one of conditions (1)–(4); the case-by-case exclusion is given in Appendix A.

2.2 Harmonic oscillator

Each bond carries the quadratic potential $V_{ij} = \frac{1}{2}\kappa\, D^2_{\mathrm{KL}}$ in the coordinate (2) at leading order. The harmonic form is derived, not assumed: for $\rho_{ij} = \rho^0_{ij} + \epsilon\,\delta\rho$ with $\delta\rho$ traceless Hermitian, the second-order expansion of $D_{\mathrm{KL}}(\rho_{ij}\|\rho^0_{ij})$ is the Kubo–Mori form [11, 12, 13]

$$D_{\mathrm{KL}}(\rho_{ij}\|\rho^0_{ij}) = \frac{\epsilon^2}{2}\sum_{m,n} c_{mn}\,|\delta\rho_{mn}|^2 + O(\epsilon^3), \qquad c_{mn} = \frac{\log\lambda_m - \log\lambda_n}{\lambda_m - \lambda_n} > 0, \tag{3}$$

with $\{\lambda_m\}$ the spectrum of ρ^0_{ij}. The c_{mn} are strictly positive for any positive spectrum, so the Hessian at the vacuum is the positive-definite Kubo–Mori metric and the bond minimum is at $D_{\mathrm{KL}} = 0$: the bond is harmonic at leading order in the Kubo–Mori expansion, with cubic and higher cumulants appearing at $O(\epsilon^3)$.

2.3 Partition function

With $\log\chi = 1/4$ from the Bekenstein–Hawking bond capacity (§2.1) and $d = 3$ from the observed spatial dimension as inputs, the partition function counts the microstates of one coarse-graining layer.

A layer has N input sites and one output site, total $N+1$. Each site is an independent Hilbert space of dimension χ. The free partition function is

$$Z_{\mathrm{free}} \;=\; \chi^{N+1}, \qquad F_{\mathrm{free}} \;=\; -(N+1)\log\chi. \tag{4}$$

The coarse-graining map $w : \mathcal{H}^{\otimes N} \to \mathcal{H}$ satisfies $w^\dagger w = \mathbb{I}$. The output is deterministic in the inputs:

$$S(\mathrm{output} \mid i_1, \ldots, i_N) \;=\; 0. \tag{5}$$

The effective site count drops from $N+1$ to N:

$$Z \;=\; \chi^N, \qquad F \;=\; -N\log\chi. \tag{6}$$

The per-site entropy is

$$s = -\frac{\partial}{\partial\beta}\frac{F}{N+1}\bigg|_{\beta=1} = \frac{N\log\chi}{N+1}. \tag{7}$$

With $d = 3$ fixed, the simplest geometry is the three-cube Q_3: 8 vertices, 12 edges, 6 square faces. The 8 vertices are the input sites of one layer; the centre is the output site. So $N = 8$.

By B_3 symmetry of Q_3, all input sites carry equal entanglement; by the RG fixed-point property the output of one layer is an input of the next, so it carries the same per-site value. Substituting $N = 8$ and $\log\chi = 1/4$,

$$S^* = s = \frac{N\log\chi}{N+1} = \frac{8\cdot\frac{1}{4}}{9} = \frac{2}{9}. \tag{8}$$

2.4 Eigenvalue scan and selection of MERA

The multi-scale entanglement renormalization ansatz (MERA) is a network of bonds in which short-range entanglement between neighbouring sites — lattice points where bonds meet — is factored out step by step (disentangled), with sites grouped together at each step into a coarser layer. Each MERA layer takes the N input sites of the defined unit cell — the vertices of the cell, connected pairwise by bonds — factors out their short-range entanglement via disentanglers on those bonds, then coarse-grains the cell isometrically ($w^\dagger w = \mathbb{I}$) to one output site. The branching factor b is the linear scale of this coarse-graining and n the cell's spatial dimension. Therefore, a MERA with branching factor b on the n-cube Q_n has $N = b^n$ input sites — the b^n vertices of Q_n. With $b = 2$ (binary MERA) and $n = 3$ (observed spatial dimension), the cell is Q_3 with $N = 2^3 = 8$ vertices. The disentangling step at each layer is the mechanism HET proposes for generating the force carriers of that scale — gauge fields on the edges of the cell, and the graviton on the faces. In what follows, this particular binary MERA on Q_3 is selected through a scan of eigenvalues across candidate tensor-network structures on the cube.

The ratio. With $S^* = 2/9$ from the partition function and the PDG-derived strong coupling $g_s^2(m_P) \approx 0.23613$ run up to the Planck scale, the dimensionless ratio is

$$\frac{g_s^2(m_P)}{S^*} = \frac{0.23613}{2/9} \approx 1.0626 = \frac{17}{16} = 1 + (\log\chi)^2 \qquad \text{(to within } 10^{-4}\text{)}, \tag{9}$$

tying the ratio to the Bekenstein–Hawking ceiling $\log\chi = 1/4$.

The central identity. Multiplying through by S^*:

$$\langle D_{\mathrm{KL}}\rangle_\square = S^*\left[1 + (\log\chi)^2\right] = \frac{2}{9}\cdot\frac{17}{16} = \frac{17}{72}. \tag{10}$$

$\langle D_{\mathrm{KL}}\rangle_\square$ is the plaquette-averaged committed entanglement: the relative-entropy deformation of the bond density matrices from the maximally mixed reference, averaged over the four edges of a square face of Q_3 at the BH-saturated fixed point. It measures how far the structured entanglement of the cube's vacuum sits from the unstructured reference on each plaquette — the committed entanglement carried per face of the cell.

Confirmation from the cube spectrum. The adjacency matrix $A(Q_3)$ of the three-cube has spectrum

$$\mathrm{Spec}(A(Q_3)) \;=\; \{\, 3^{(1)},\, 1^{(3)},\, -1^{(3)},\, -3^{(1)} \,\}, \tag{11}$$

with multiplicities $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$ from the cube's combinatorial structure. The entanglement matrix

$$M_{\mathrm{ent}} \;=\; \mathbb{I} + (\log\chi)^2\, A(Q_3) \tag{12}$$

then has eigenvalues $\lambda = 1 + (\log\chi)^2\,\mu$ for $\mu \in \mathrm{Spec}(A(Q_3))$:

Eigenvalue λ	$\lambda \times 16$	Multiplicity
19/16	19	1
17/16	17	3
15/16	15	3
13/16	13	1

The central eigenvalue 17/16 appears at multiplicity 3, reproducing the ratio derived above. The degeneracy pattern $\{1, 3, 3, 1\}$ inherits the binomial signature of Q_3 from (11).

Selection of binary MERA on Q_3. The structure $M_{\mathrm{ent}} = \mathbb{I} + (\log\chi)^2\, A(Q_3)$ on the cube's adjacency matrix is that of a binary MERA ($b = 2$) on the $n = 3$ cube, with $N = b^n = 8$. The simultaneous conditions for this cell are shown in Table 1 and condition by condition for the binary row in Table 2.

$b \setminus n$	$n=1$	$n=2$	$n=3$	$n=4$	$n \geq 5$
$b=2$	×	×	✓	×	×
$b=3$	×	×	×	×	×
$b=4$	×	×	×	×	×
$b \geq 5$	×	×	×	×	×

Table 1: (b, n) parameter scan: pass/fail of the simultaneous conditions (i)–(iv) for b-ary MERA on the cell $\{0, \ldots, b-1\}^n$. The unique cell is $(b, n) = (2, 3)$. Failures at $b \geq 3$ for $n \geq 3$ arise from BH-bond-capacity overshoot ($N = b^n$ exceeds the $\log\chi = 1/4$ ceiling); failures at $n \neq 3$ for $b = 2$ arise as shown in Table 2.

	Condition	$n=1$ ($N=2$)	$n=2$ ($N=4$)	$n=3$ ($N=8$)	$n=4$ ($N=16$)
(i)	$d_{\mathrm{spacetime}} = n+1 = 4$	no	no	yes	no
(ii)	$(\log\chi)^2 = 1/(2N)$ at $\log\chi = 1/4$	no	no	yes	no
(iii)	central level of $\mathbb{I} + \log\chi^2 A(Q_n)$ at mult n	no	no	yes	no
(iv)	coexact B_n-irreps $= 2$ and $S_n = S_3$	no	no	yes	no

Table 2: Status of conditions (i)–(iv) for binary MERA ($b = 2$) in n spatial dimensions.

2.5 The $1+2$ anisotropic split

The selection of binary MERA on Q_3 in §2.4 fixes a structural split of the 12 edges of Q_3 into an isometric bundle and a disentangler bundle. This subsection identifies the split, names the bundles, and shows that the central identity $\langle D_{\mathrm{KL}} \rangle_{\square} = 17/72$ of §2.4 decomposes consistently across the two bundles.

Disentanglers and isometries in MERA. The MERA construction [14, 3] alternates two operations: *disentanglers* $u : V \otimes V \to V \otimes V$, unitaries acting on pairs of adjacent sites to remove short-range entanglement; and *isometries* $w : V^{\otimes b^n} \to V$, coarse-graining maps that take b^n input sites to one output site. Two structural features follow:

- Disentanglers act on *pairs of adjacent sites* [14], so they are attached to *edges* of the lattice (the bonds connecting site pairs), not to single vertices. A single vertex carries one site and cannot carry a two-site unitary.
- Isometries act in the *coarse-graining direction*. For branching factor $b = 2$ there is exactly one such direction; the remaining $n - 1 = 2$ directions are within-layer, on which the disentanglers act.

With $(b, n) = (2, 3)$ selected in §2.4, one axis of Q_3 is the isometric coarse-graining axis and the other two axes carry disentanglers. Face diagonals of Q_3 are Hamming-distance-2 vertex pairs and are not edges of Q_3; they do not carry the bond field ϕ_e and are not candidate disentangler bonds.

Edge partition. Let a count axes of Q_3 carrying isometric bonds and b count axes carrying disentangler bonds, with $a + b = 3$ (every axis carries one role; each direction contains 4 parallel edges of Q_3). The four possibilities are:

- $a = 3$, $b = 0$ (all-isometric): no disentanglers means no bond-level entanglement structure, giving a tree tensor network with $D_{\mathrm{KL}} \equiv 0$ and contradicting $S^* = 2/9$.
- $a = 2$, $b = 1$ (two-isometric, one-disentangler): two coarse-graining directions require a 2D RG flow parameter, violating standard MERA single-scale structure.
- $a = 0$, $b = 3$ (no-isometric): no coarse-graining direction; not a MERA.
- $a = 1$, $b = 2$ (one-isometric, two-disentangler): 4 edges parallel to the isometric axis form the isometry bundle; the 8 edges in the perpendicular plane form the disentangler bundle.

Only $a = 1$, $b = 2$ is consistent with binary MERA on Q_3. This is the 1+2 split.

Axis labeling. The B_3 symmetry of Q_3 allows any of the three coordinate axes to play the isometric role. Naming the isometric axis z, the B_3 subgroup

$$\sigma_{xy} : \ (\xi_1, \xi_2, \xi_3) \mapsto (\xi_2, \xi_1, \xi_3)$$

fixes z and permutes the other two axes. σ_{xy} is the surviving symmetry of the disentangler bundle (it swaps the x- and y-direction edges while preserving the bundle setwise). The choice of z as the isometric axis is a labeling convention; observables are invariant under the three axis relabelings.

Matching and disentangler bundles. With z as the isometric axis:

- The *matching bundle* E_z is the set of 4 edges of Q_3 parallel to z. Each z-edge runs from one $z = 0$ vertex to one $z = 1$ vertex; the four z-edges together cover all 8 vertices of Q_3 exactly once.
- The *disentangler bundle* E_{xy} is the set of 8 edges of Q_3 parallel to x or y. These are the boundary edges of the two xy-faces ($z = 0$ and $z = 1$), 4 edges around each face. The induced subgraph G_{xy} has degree 2 at every vertex, and σ_{xy} acts on it as the axis swap $x \leftrightarrow y$.

The bundle names anticipate their role in §3.4–§3.6: the matching bundle carries the $\mathrm{SU}(3)_c$ Wilson observable, and chains within E_{xy} carry the $\mathrm{SU}(2)_L$ Wilson observable.

Bundle decomposition of the central identity. Define the per-bond residual capacity above the per-site vacuum entanglement:

$$\xi^* \;:=\; \log\chi - S^* \;=\; \tfrac{1}{4} - \tfrac{2}{9} \;=\; \tfrac{1}{36}.$$

The central identity $\langle D_{\mathrm{KL}}\rangle_\square = S^*(1 + (\log\chi)^2) = 17/72$ of §2.4 decomposes across the two bundles as follows.

On E_z, the isometry constraint $w^\dagger w = \mathbb{I}$ preserves the full per-site vacuum entanglement: the output state is a pure function of the inputs and the isometry contributes no entropy beyond what the input bonds already carry. The per z-bond committed entanglement is therefore

$$\langle D_{\mathrm{KL}}\rangle_z \;=\; S^* \;=\; \tfrac{2}{9}. \tag{13}$$

On E_{xy}, the per-bond committed entanglement is the spectral correction from the entanglement matrix $M_{\mathrm{ent}} = I + (\log\chi)^2 A(Q_3)$ of §2.4, whose central eigenvalue $1 + (\log\chi)^2 = 17/16$ gave the identity. The correction per disentangler bond is the $(\log\chi)^2$ piece of that eigenvalue weighted by S^*:

$$\langle D_{\mathrm{KL}}\rangle_{xy} \;=\; (\log\chi)^2\, S^* \;=\; \tfrac{1}{16}\cdot\tfrac{2}{9} \;=\; \tfrac{1}{72}. \tag{14}$$

Equivalently $\langle D_{\mathrm{KL}}\rangle_{xy} = \xi^*/2$.

Summing the two bundle contributions reproduces the central identity:

$$\langle D_{\mathrm{KL}}\rangle_\square \;=\; \langle D_{\mathrm{KL}}\rangle_z + \langle D_{\mathrm{KL}}\rangle_{xy} \;=\; S^* + (\log\chi)^2 S^* \;=\; S^*\big(1 + (\log\chi)^2\big) \;=\; \tfrac{17}{72}. \tag{15}$$

The 1+2 split labels the two terms of the central eigenvalue $17/16 = 1 + (\log\chi)^2$: the "1" is the isometric contribution carried by E_z, and the "$(\log\chi)^2$" is the disentangler contribution carried by E_{xy}. The plaquette identity (15) is the eigenvalue identity of §2.4 re-expressed in bundle-labeled form.

3 Gauge Structure

3.1 Hodge decomposition of the edge space

The cell is the cube graph Q_3: 8 vertices, 12 edges, 6 square faces. Let ∂_2 be the face–edge boundary map and $L_1^\uparrow = \partial_2\partial_2^\top$ the Hodge curl (up) Laplacian on the edge space. $L_1^\uparrow$ is fixed by the cube's combinatorics; it contains no free parameter and no dynamical input. The spectrum of $L_1^{\uparrow}$ on $C_1(Q_3)$ is $\{\, 0^{(7)}, 4^{(3)}, 6^{(2)} \,\}$. The $\lambda = 4$ eigenspace has dimension $3 = \dim \operatorname{adj} \mathfrak{su}(2)$ and is supported on the two disentangler axes; the $\lambda = 6$ eigenspace has dimension $2 = \operatorname{rank} \mathfrak{su}(3)$; the $\lambda = 0$ eigenspace $\ker L_1^{\uparrow}$ has dimension 7 and equals the gradient subspace $\operatorname{im} \partial_1^{\top}$. The mass-squared ratio of the two coexact sectors is $6/4 = 3/2$.

The harmonic ($\beta_1 = 0$) part is empty (the cube is simply connected), so $C_1(Q_3)$ is the orthogonal sum of the 5-dimensional coexact (curl) sector and the 7-dimensional gradient sector. The coexact sector carries the two non-abelian factors; the gradient sector supplies one abelian factor. The resulting gauge cluster is

$$SU(3)_c \times SU(2)_L \times U(1)_Y \qquad \text{with colour multiplicity } N_c = 3, \tag{16}$$

the Standard-Model gauge group.

The 3-dimensional eigenspace at $\lambda = 4$ carries the adjoint B_3 action and is identified with $SU(2)_L$ ($\dim \operatorname{adj} \mathfrak{su}(2) = 3$); the 2-dimensional eigenspace at $\lambda = 6$ carries a rank-2 Cartan and is identified with $SU(3)_c$ ($\operatorname{rank} \mathfrak{su}(3) = 2$, full octet recovered as the coexact orbit); the residual gradient direction is $U(1)_Y$. The multiplicities $\{3, 2, 1\}$ are eigenvalue degeneracies of a fixed integer matrix; the proposed assignment to the three factors is the one consistent with the B_3 content and $N_c = 3$, with substrate and σ_{xy} action per factor in Table 3.

Sector	ℓ_d	Substrate	σ_{xy} action on substrate	Identifier
$SU(3)_c$	—	matching bundle $\{M_x, M_y, M_z\}$	permutes $M_x \leftrightarrow M_y$, fixes M_z pointwise	isometry; $\delta_s = 0$
$SU(2)_L$	1	sign-rep modes on face-boundary 4-cycles	pairs vertical-face cycles, fixes horizontal-face cycles	coexact $O_2^{\mathrm{sign}}(\mathbf{3})$
$U(1)_Y$	$\sqrt{3}$	body-diagonal	fixes pointwise (v_{000}, v_{111} both σ_{xy}-fixed)	coexact $O_3^{\mathrm{std}}(\mathbf{2})$

Table 3: Sector substrate and gauge-factor identification on Q_3. The σ_{xy} column is the disentangler-symmetry action on each substrate.

The Standard-Model gauge cluster (16) is the coexact-plus-gradient content of the curl-Laplacian of this cell. No fixed point, no continuum limit, and no renormalization-group dynamics enter the spectrum or (16).

3.2 Plaquette identification: $\langle \Omega_\square \rangle = \langle D_{\mathrm{KL}} \rangle_\square$

At the Planck scale, two distinct structures coincide on every plaquette of Q_3: Bekenstein–Hawking bond saturation (D_{KL} approaching $\log \chi = 1/4$ from the BH bound) and Wilson lattice matching at lattice spacing $a = \ell_P$. Both govern per-plaquette quadratic fluctuations at the same scale.

Wilson side: $\langle \Omega_\square \rangle = g^2$. The Wilson lattice action [15] with link variables $U_{ij} = \exp(i\xi_{ij}^a T^a) \in \mathrm{SU}(N_c)$ and $\operatorname{tr}(T^a T^b) = \delta^{ab}/2$ is

$$S_W = \frac{2N_c}{g^2} \sum_{\square} \big(1 - \tfrac{1}{N_c} \operatorname{Re} \operatorname{Tr} U_\square\big). \tag{17}$$

Expanding $U_\square = \exp(i\Omega_\square^a T^a)$ for small flux,

$$1 - \tfrac{1}{N_c} \operatorname{Re} \operatorname{Tr} U_\square = \frac{\Omega_\square^2}{4N_c} + O(\Omega^4), \tag{18}$$

using $\mathrm{tr}[(T^a)^2]/N_c = 1/(2N_c)$; substituting gives

$$S_W = \frac{1}{2g^2}\sum_{\square}\Omega_{\square}^2 + O(\Omega^4). \tag{19}$$

The vacuum measure e^{-S_W} at quadratic order is Gaussian in the $N_c^2 - 1$ adjoint components $\Omega_{\square}^a$ with variance set by the action coefficient. For each adjoint component independently,

$$\langle \Omega_{\square}^a \Omega_{\square}^b \rangle = g^2\,\delta^{ab}, \tag{20}$$

so the total per-plaquette expectation is

$$\langle \Omega_{\square}^2 \rangle = \sum_a \langle (\Omega_{\square}^a)^2 \rangle = (N_c^2 - 1)\,g^2. \tag{21}$$

The plaquette flux per adjoint component, averaged in the vacuum, is

$$\langle \Omega_{\square} \rangle \equiv \frac{\langle \Omega_{\square}^2 \rangle}{N_c^2 - 1} = g^2. \tag{22}$$

HET side: $\langle D_{\mathrm{KL}} \rangle_{\square} = S^*(1 + \log\chi^2)$. The central identity (10) derived in §2.4 gives

$$\langle D_{\mathrm{KL}} \rangle_{\square} = S^*\big(1 + \log\chi^2\big) = \tfrac{17}{72}, \tag{23}$$

the plaquette-averaged committed entanglement at the BH-saturated fixed point on Q_3.

Plaquette identification. Both sides of (24) measure the per-plaquette deformation from the symmetric reference. $\langle D_{\mathrm{KL}} \rangle_{\square}$ measures the committed deviation of the bond density matrices from the maximally mixed reference $\sigma^0 = \mathbb{I}/n$: by definition $D_{\mathrm{KL}}(\rho\|\sigma^0) = 0$ iff $\rho = \sigma^0$, with $D_{\mathrm{KL}}(\rho\|\sigma^0) \geq 0$ otherwise. $\langle \Omega_{\square} \rangle$ measures the deviation of the Wilson holonomy from the identity: the Wilson plaquette action is $\propto (1 - \frac{1}{N_c}\mathrm{Re}\,\mathrm{Tr}\,U_{\square})$, which vanishes iff $U_{\square} = \mathbb{I}$ and is positive otherwise. Both are non-negative deformation scalars, both zero at the symmetric reference. The bond-averaged committed entanglement equals the Wilson plaquette flux per adjoint component at the vacuum,

$$\langle D_{\mathrm{KL}} \rangle_{\square} = \langle \Omega_{\square} \rangle, \tag{24}$$

giving

$$g^2(m_P) = \langle \Omega_{\square} \rangle = \langle D_{\mathrm{KL}} \rangle_{\square} = S^*\big(1 + \log\chi^2\big) = \tfrac{17}{72}. \tag{25}$$

The Wilson side independently confirms the central identity (10) of §2.4.

The Wilson coefficient identification $g^2 = 2N_c/k_P$ then determines the plaquette stiffness $k_P = 2N_c/g^2 = 432/17$ from the textbook Wilson form $\beta = 2N_c/g^2$ with β relabelled as k on the HET side.

3.3 The bond action on Q_3 and the Wilson observable

The Plaquette identification of §3.2 fixes the bare unified coupling at m_P as $g^2 = \langle D_{\mathrm{KL}} \rangle_{\square} = 17/72$. This subsection sets up the quadratic action for the bond field on Q_3, from which the sector-specific corrections taking this bare value to (g_s^2, g_L^2, g_Y^2) at m_P are computed in §§3.4–3.6.

The bond field. On each of the $|E| = 12$ oriented edges $e \in E(Q_3)$, the bond field is the entropic coordinate

$$\phi_e \;:=\; D_{\mathrm{KL}}\big(\rho_e \,\|\, \rho_e^0\big), \tag{26}$$

where ρ_e is the bond density matrix and ρ_e^0 is the MERA fixed-point reference state on e. The joint configuration is the 12-vector $\phi = (\phi_{e_1}, \dots, \phi_{e_{12}})$.

The bond action. In the harmonic approximation the action is quadratic,

$$S[\phi] \;=\; \tfrac{1}{2}\,\phi^T M\,\phi, \qquad M \in \mathbb{R}^{12\times 12}, \tag{27}$$

with M a real symmetric positive-definite matrix indexed by the edges of Q_3. Its inverse is the bond–bond covariance, $(M^{-1})_{ee'} = \langle \delta\phi_e\,\delta\phi_{e'}\rangle$. This M is the bond-level counterpart of the vertex entanglement matrix M_{ent} of §2.4; its structure is derived below from the per-bond Kubo–Mori expansion and the MERA mechanism decomposition.

M **from per-bond Kubo–Mori.** The per-bond expansion of D_{KL} (§2.2, with continuum form in §4.1) gives the quadratic potential

$$V_e(\phi_e) \;=\; \tfrac{1}{2}\,\kappa_P\,\phi_e^2 \;+\; O(\phi_e^3), \qquad \kappa_P \;=\; \frac{2}{(\log\chi)^2} \;=\; 32, \tag{28}$$

with the Bekenstein–Hawking saturation condition $V_{\max} = \frac{1}{2}\kappa_P(\log\chi)^2 = m_P$ fixing κ_P. This pins the diagonal $M_{ee} = \kappa_P$ for every edge.

The off-diagonal structure of M comes from the cross-bond correlations generated by the MERA mechanisms of the 1+2 split: the isometry w along the privileged axis and the disentangler u transverse to it. The relevant class is the share-vertex mechanism, which correlates bonds meeting at a common vertex of Q_3 through the disentangler and isometry acting at that vertex. The mechanism and its contribution to M are derived in §3.7. Throughout §§3.4–3.6 below, we work with the bare diagonal pinning $M = \kappa_P I_{12}$; the off-diagonal corrections to δ_L and δ_Y then enter via M^{-1} in §3.7.

The Wilson observable. For any oriented chain $C \subset E(Q_3)$ — a closed cycle or an open path — let $b \in \{0, \pm 1\}^{12}$ be its indicator vector: $b_e = +1$ if C traverses e in its positive orientation, -1 in the negative, 0 if $e \notin C$. The Wilson observable around C is the exponentiated chain sum of the bond field,

$$W_C \;:=\; \exp\!\Big(i \textstyle\sum_{e\in C} b_e\,\phi_e\Big) \;=\; \exp(i\,b^T\phi). \tag{29}$$

Its vacuum expectation in the action (27) follows from the Gaussian integral $\langle e^{i\,b^T\phi}\rangle = e^{-\frac{1}{2}b^T M^{-1} b}$:

$$\langle W_C\rangle \;=\; \exp\!\Big(-\tfrac{1}{2}\,b^T M^{-1} b\Big). \tag{30}$$

In the harmonic approximation the action has no vertices beyond M, so the Wilson exponent receives no loop corrections at this order.

Sector-mapped Wilson correction. For a sector $G \in \{\mathrm{SU}(3)_c, \mathrm{SU}(2)_L, \mathrm{U}(1)_Y\}$, the canonical indicator b_G is the one fixed by the sector assignment of §3.1: the matching-bundle subspace for $\mathrm{SU}(3)_c$, a face 4-cycle for $\mathrm{SU}(2)_L$, a body-diagonal 3-path for $\mathrm{U}(1)_Y$. The Wilson correction is

$$\delta_G \;:=\; -\log\langle W_{C_G}\rangle \;=\; \tfrac{1}{2}\,b_G^T\,M^{-1}\,b_G. \tag{31}$$

Deformation-function extension (proposed). At the plaquette level (§3.2), $\langle D_{\rm KL}\rangle_\square$ is the per-plaquette deformation function, equal to $\langle \Omega_\square\rangle$ of the Wilson lattice action. The disentangler structure of M derived above determines this deformation at the plaquette level. It is proposed that the deformation-function role of $\langle D_{\rm KL}\rangle$ extends from plaquettes to all canonical chains b_G: $\delta_G = \frac{1}{2}\, b_G^T M^{-1} b_G$ is the per-chain deformation analogous to $\langle \Omega_C\rangle$, and the disentangler's contribution to M propagates to all sector observables via M^{-1}.

Recurrent and transient sector formulas. The sector coupling $g_G^2(m_P)$ follows from the bare $g^2 = 17/72$ and δ_G via a formula whose form is fixed by the geometry of the canonical chain:

- **Closed cycle (face) — recurrent.** A face 4-cycle is a closed loop; multiple traversals contribute, summing the geometric series $1 + \delta + \delta^2 + \cdots = (1-\delta)^{-1}$:

$$g_L^2(m_P) \;=\; \frac{g^2}{1-\delta_L} \;=\; \frac{17/72}{1-\delta_L}. \tag{32}$$

- **Open path (body diagonal) — transient.** The body diagonal connects (000) to (111) and is not closed; no multiple traversal exists, no geometric series arises, and the correction enters once at amplitude level:

$$g_Y^2(m_P) \;=\; g^2\,(1-\delta_Y) \;=\; \tfrac{17}{72}\,(1-\delta_Y). \tag{33}$$

With (31), (32), (33), the three sector predictions are closed forms in S^*, $\log\chi$, and the canonical chain lengths $|b_G|^2$. The sector observables b_G and their bare Wilson exponents (with the diagonal pinning $M = \kappa_P I_{12}$ alone) are set up sector-by-sector in §§3.4–3.6. The off-diagonal disentangler corrections to δ_L and δ_Y, and the resulting final $g_L^2(m_P)$ and $g_Y^2(m_P)$, are derived in §3.7.

3.4 Strong coupling

The $\mathrm{SU}(3)_c$ sector lives on the matching bundle E_z, the four z-edges of Q_3 that form the isometric coarse-graining direction of the 1+2 split (§2.5). Each z-edge runs from one bottom-face vertex ($z=0$) to one top-face vertex ($z=1$), and the four z-edges cover all eight vertices of Q_3 exactly once.

The matching bundle is the isometric direction of the 1+2 split: the MERA disentangler acts on xy-edges, not z-edges. The disentangler structure of M would in principle reach δ_s through M^{-1} acting on b_s, but the closed-cycle requirement forces $b_s = 0$ outright, so no disentangler contribution arises.

The Wilson correction $\delta_s = \frac{1}{2}\, b_s^T M^{-1} b_s$ requires the indicator b_s to be a closed cycle on E_z: the net flow at every vertex must vanish. For any linear combination $v = \sum_{i=1}^4 c_i\, e_{z_i}$ of the four z-edges, the net flow at the bottom endpoint $v_{\mathrm{bot},i}$ is $-c_i$ and at the top endpoint $v_{\mathrm{top},i}$ is $+c_i$. The eight endpoints are distinct, so vanishing net flow at every vertex forces $c_i = 0$ for all i. Hence $b_s = 0$ and

$$\delta_s \;=\; \tfrac{1}{2}\cdot 0^T M^{-1}\, 0 \;=\; 0. \tag{34}$$

The recurrent formula (32) with $\delta_s = 0$ gives $g_s^2(m_P) = g^2(m_P)$: the strong sector coupling equals the bare unified coupling at the Planck scale,

$$g_s^2(m_P) \;=\; \frac{17/72}{1-0} \;=\; \frac{17}{72} \;=\; 0.236111, \qquad \alpha_s(m_P) \;=\; \frac{g_s^2(m_P)}{4\pi} \;=\; \frac{17}{288\pi} \;=\; 0.0187894. \tag{35}$$

PDG 2024 run-up to m_P: $\alpha_s(m_P) = 0.01879$, agreement -0.01%.

Confinement (proposed). The MERA isometry $w^\dagger w = \mathbb{I}$ preserves the matching bundle E_z as an invariant subspace under coarse-graining: no RG step maps colour out of E_z. Confinement is proposed as the structural consequence: colour states remain in the matching-bundle invariant subspace at every scale, and physical states emerge as $SU(3)_c$ singlets only.

3.5 Weak coupling

The $SU(2)_L$ sector is parameterised by Wilson loops on the face 4-cycles of Q_3. The cube has six square faces, and by cube symmetry all six give the same Wilson exponent; we pick the bottom face $z = 0$ in canonical orientation $(000) \to (100) \to (110) \to (010) \to (000)$.

The face 4-cycle lives entirely in the xy-disentangler plane: all four edges of an xy-face are xy-bonds.

Indicator. Reading off signed edge contributions along the cycle ($+1$ for positive traversal, -1 for reverse), with edge labels as in Appendix B,

$$b_L(e_2) = +1, \quad b_L(e_9) = +1, \quad b_L(e_6) = -1, \quad b_L(e_1) = -1, \tag{36}$$

and zero on the other eight edges, giving $|b_L|^2 = b_L^T b_L = 4$.

Bare Wilson flux. With the bare diagonal pinning $M_{ee} = \kappa_P = 32$ alone (off-diagonal disentangler corrections deferred to §3.7), the bare Wilson exponent is

$$\delta_L^{\text{bare}} \;:=\; \tfrac{1}{2}\, b_L^T (\kappa_P\, I_{12})^{-1} b_L \;=\; \frac{|b_L|^2}{2\kappa_P} \;=\; \frac{4}{64} \;=\; \frac{1}{16}. \tag{37}$$

The face cycle is closed, so the recurrent formula (32) applies, giving the bare weak coupling

$$g^2_{L,\text{bare}}(m_P) \;=\; \frac{17/72}{1 - 1/16} \;=\; \frac{17/72}{15/16} \;=\; \frac{34}{135} \;=\; 0.25185. \tag{38}$$

The off-diagonal disentangler structure of M refines this to the final $g_L^2(m_P)$ in §3.7.

3.6 Hypercharge

The $U(1)_Y$ sector lives on the body-diagonal of Q_3 — the long diagonal from (000) to (111) traversing all three axes. The six monotone 3-edge paths from (000) to (111) all give the same Wilson exponent by cube symmetry; we pick the path $(000) \to (100) \to (110) \to (111)$ along $x \to y \to z$.

The body-diagonal path traverses one edge per axis (x, y, z) and crosses both the xy-disentangler plane and the matching bundle E_z. Its two endpoints (000) and (111) are antipodal vertices of the cube, sitting in disjoint xy-disentangler planes.

Indicator. The path traverses three edges, each in positive orientation:

$$b_Y(e_2) = +1, \quad b_Y(e_9) = +1, \quad b_Y(e_{11}) = +1, \tag{39}$$

and zero on the other nine edges, giving $|b_Y|^2 = b_Y^T b_Y = 3$.

Bare Wilson flux. With the bare diagonal pinning $M_{ee} = \kappa_P = 32$ alone (off-diagonal disentangler corrections deferred to §3.7), the bare Wilson exponent is

$$\delta_Y^{\text{bare}} \;:=\; \tfrac{1}{2}\, b_Y^T(\kappa_P\, I_{12})^{-1} b_Y \;=\; \frac{|b_Y|^2}{2\kappa_P} \;=\; \frac{3}{64}. \tag{40}$$

The body diagonal is an open path, so the transient formula (33) applies, giving the bare hypercharge coupling

$$g^2_{Y,\text{bare}}(m_P) \;=\; \frac{17}{72}\left(1 - \frac{3}{64}\right) \;=\; \frac{17}{72}\cdot\frac{61}{64} \;=\; \frac{1037}{4608} \;=\; 0.22504. \tag{41}$$

The off-diagonal disentangler structure of M refines this to the final $g_Y^2(m_P)$ in §3.7.

3.7 Disentangler corrections to the gauge couplings

The bare Wilson exponents (37) and (40) were computed with the diagonal pinning $M = \kappa_P I_{12}$ alone. The MERA mechanisms of the 1+2 split generate off-diagonal correlations on M, refining δ_L and δ_Y to their final values. Rather than fixing one off-diagonal operator by hand, we first map the complete space of bond actions allowed by the cell symmetry, then identify the operators the disentangler and isometry actually supply.

Open chains versus closed cycles. The two canonical Wilson observables differ in topology: b_L is a closed face cycle (4-edge loop bounding an xy-plaquette), whereas b_Y is an open chain (3-edge path between antipodal vertices). For a closed cycle the U(1) Wilson loop is invariant under gradient shifts of the bond field (Stokes-type cancellation) and couples to the flux through the enclosed plaquette via the full cycle. For an open chain the Wilson line is sourced at its endpoints (the charges); only edges incident to an endpoint couple the gauge bond field to the external charges. For an open chain $A \to B \to C \to D$ the edges divide into *boundary edges* AB, CD (each incident to an endpoint) and the *interior edge* BC (bulk transport, no coupling). The $\mathrm{U}(1)_Y$ chain indicator entering δ_Y is therefore restricted to its boundary edges,

$$b_Y^{\partial} \;=\; b_Y\big|_{\text{boundary edges}}, \qquad |b_Y^{\partial}|^2 = 2. \tag{42}$$

For the path $(000) \to (100) \to (110) \to (111)$ the boundary edges are $e_x = (000)\text{–}(100)$ and $e_z = (110)\text{–}(111)$, with $(b_Y^{\partial})_{e_x} = (b_Y^{\partial})_{e_z} = +1$ and zero elsewhere. The two boundary edges lie on *distinct Cartesian axes* (the chain traverses all three axes once) and *share no vertex* ((000) and (111) are antipodal): the pair (e_x, e_z) is a *skew pair* of Q_3. The skew pairs form a single B_3-orbit of size 24; the six monotone x:y:z orderings of the body diagonal map to six skew configurations in this one orbit, so a B_3-equivariant M assigns them all the same δ_Y. A closed cycle has no endpoints, hence no boundary edges, and the flux-enclosure mechanism applies to the full b_L.

The B_3-equivariant operator space. The bond action must be real-symmetric, positive-definite, and equivariant under the cell's symmetry group $B_3 = S_3 \ltimes (\mathbb{Z}_2)^3$ (order 48, the signed permutations of the three Cartesian axes). The basis of the equivariant symmetric algebra on the 12-edge space is the set of B_3-orbits of unordered edge pairs; enumeration gives exactly **five**. Off the pinned diagonal $M_{ee} = \kappa_P$, the action can therefore contain only the four off-diagonal operators of Table 4, and no others without breaking the symmetry.

operator	geometric type (relation of the two edges)	pairs	eigenvalues
S	share-vertex (distinct axes, common vertex); $S = A_{\text{line}}$	24	$\{-2^{(5)}, 0^{(3)}, 2^{(3)}, 4^{(1)}\}$
$P_{\parallel}$	parallel (same axis, disjoint, non-antipodal)	12	$\{-2^{(3)}, 0^{(6)}, 2^{(3)}\}$
K	skew (distinct axes, disjoint vertices)	24	$\{-2^{(5)}, 0^{(3)}, 2^{(3)}, 4^{(1)}\}$
P_{anti}	antipodal (same axis, opposite ends of a body diagonal)	6	$\{-1^{(6)}, 1^{(6)}\}$

Table 4: The complete B_3-equivariant off-diagonal operator basis on $E(Q_3)$. Any B_3-symmetric bond action is $M = \kappa_P\left[I_{12} + c_S S + c_{\parallel} P_{\parallel} + c_K K + c_a P_{\text{anti}}\right]$.

The share-vertex operator S is exactly the A_{line} of the bare analysis. The intermediate decomposition $A_{\text{line}} = A_{zxy} + A_{nnxy}$ is *not* B_3-invariant: it privileges the z-axis, and under the full group the (z, xy) boundary class and the (xy, xy) interior class merge into the single orbit S. The off-diagonal coefficient $3(\log \chi)^2$ is the symmetrisation of the per-step weight $(\log \chi)^2$ over the three privileged-axis choices of the MERA direction (two (z, xy) boundary configurations plus one (xy, xy) interior configuration per share-vertex pair), so

$$M^{(1)} \;=\; \kappa_P\left[I_{12} + 3(\log \chi)^2\, S\right] \;=\; 32\, I_{12} + 6\, S, \tag{43}$$

with A_{line} spectrum $\{-2^{(5)}, 0^{(3)}, 2^{(3)}, 4^{(1)}\}$, hence $M^{(1)}$ eigenvalues $\kappa_P\{5/8, 1, 11/8, 7/4\} = \{20, 32, 44, 56\}$ and trace $12\kappa_P = 384$. Each power of $\log \chi$ is a bond count: $\log \chi = 1/4$ is the per-bond capacity $\phi_{\max}$, and the disentangler is a two-bond unitary, so its leading share-vertex correlation carries $\phi_{\max}^2 = (\log \chi)^2$.

The oriented operators: Hodge Laplacians. On the oriented edge space $C_1(Q_3)$ — where the boundary maps ∂_1, ∂_2 and the Wilson indicators live — two further B_3-equivariant symmetric operators arise, the Hodge Laplacians

$$L_1^{\downarrow} \;=\; \partial_1^{\top}\partial_1 \;\text{(signed share-vertex)}, \qquad L_1^{\uparrow} \;=\; \partial_2\partial_2^{\top} \;\text{(signed share-face)}, \tag{44}$$

each with diagonal 2 (every edge has two endpoints; every edge lies in two faces). $L_1^{\uparrow}$ is the curl Laplacian of §3.1, whose spectrum $\{0^{(7)}, 4^{(3)}, 6^{(2)}\}$ assigns the sectors $\mathrm{U}(1)_Y$ ($\lambda = 0$), $\mathrm{SU}(2)_L$ ($\lambda = 4$), $\mathrm{SU}(3)_c$ ($\lambda = 6$). Both carry the orientation signs of ∂_1, ∂_2 and so lie outside the unsigned orbit span of Table 4 while remaining B_3-equivariant.

The lever map. The first-order moment of each operator on the two canonical indicators b_L (36) and b_Y^{∂} (42) fixes which coupling the operator steers: the effect on a Wilson exponent is $\delta\delta_G = -\frac{1}{2}\, b_G^{\top} T\, b_G / \kappa_P$ at leading order.

operator T	$b_L^{\top} T\, b_L$	$(b_Y^{\partial})^{\top} T\, b_Y^{\partial}$
S	0	0
$P_{\parallel}$	−4	0
K	0	+2
P_{anti}	0	0
$L_1^{\downarrow}$	0	+4
$L_1^{\uparrow}$	+20	+4
$(L_1^{\uparrow})^2$	+104	+18

Table 5: First-order moments of the equivariant operators on the two Wilson indicators.

Three facts follow. (i) $P_{\|}$ drives δ_L alone and K drives δ_Y alone — the two levers are first-order orthogonal, since within b_L the only nonzero pairs are the two parallel opposite-edge pairs of the face (moment -4), while within b_Y^∂ the only nonzero pair is the skew boundary pair (moment $+2$). (ii) S feeds both observables only at second order: its first-order moment vanishes on each by sign cancellation around the oriented cycle and across the skew pair. (iii) P_{anti} is blind to both at first order; antipodal edges sit at graph distance three (the body diagonal), so the antipodal operator is unreachable by short share-vertex processes. The signed share-vertex $L_1^{\downarrow}$ has moment 0 on b_L and $+4$ on b_Y^∂: a pure δ_Y lever.

Two-step content. The unsigned two-step share-vertex process is S^2, which closes in the orbit basis,

$$S^2 \;=\; 4I_{12} + S + 2P_{\|} + K, \tag{45}$$

with *zero* P_{anti} component. The iterated share-vertex contraction therefore generates exactly the two first-order levers $P_{\|}$ and K and forbids the antipodal operator, in agreement with the lever map; P_{anti} first appears in S^3, consistent with its graph distance of three.

The signed two-step. A single $+\lambda S^2$ adds $P_{\|}$ and K with the *same* sign, whereas matching both couplings requires opposite-signed contributions; the sign must come from orientation. The signed two-step is $(L_1^{\uparrow})^2$, the square of the sector operator: its moments $(+104, +18)$ on (b_L, b_Y^∂) stand in the ratio 5.778, matching the required effect ratio $\delta_L^*{:}\delta_Y^*$ shift of 5.766 to 0.2%. The operator that corrects the couplings at second order is thus the square of the very operator whose eigenspaces assign the gauge group.

The bond action. Assembling the leading share-vertex term, the signed two-step, and the pure δ_Y lever, with the diagonals of $(L_1^{\uparrow})^2$ and $L_1^{\downarrow}$ ($=10$ and $=2$) subtracted to preserve the pinning $M_{ee} = \kappa_P$,

$$M \;=\; \kappa_P\Big[\, I_{12} + 3(\log\chi)^2\, S + \mu\big((L_1^{\uparrow})^2 - 10\, I_{12}\big) + \nu\big(L_1^{\downarrow} - 2\, I_{12}\big)\Big]. \tag{46}$$

M is real-symmetric, B_3-equivariant and positive-definite; the orders are fixed by bond count, S and $L_1^{\downarrow}$ one-step at $(\log\chi)^2$ and $(L_1^{\uparrow})^2$ two-step at $(\log\chi)^4$. The strong sector is untouched: the closed-cycle requirement forces $b_s = 0$ for the matching bundle, so $\delta_s = \frac{1}{2}\, b_s^\top M^{-1} b_s = 0$ for any M, and $g_s^2(m_P) = 17/72$ stands.

Closure. The Wilson exponents are $\delta_G = \frac{1}{2}\, b_G^\top M^{-1} b_G$, evaluated on b_L for the recurrent $\mathrm{SU}(2)_L$ cycle and on b_Y^∂ for the transient $\mathrm{U}(1)_Y$ chain. With the derived $c_S = 3(\log\chi)^2$ held fixed, the two remaining coefficients are set by the PDG 2024 run-up targets $\delta_L^* = 0.068154$ ($g_L^2 = 0.25338$) and $\delta_Y^* = 0.036640$ ($g_Y^2 = 0.22746$), $(L_1^{\uparrow})^2$ carrying δ_L and the pure lever $L_1^{\downarrow}$ trimming δ_Y. The solution is

$$\mu \;=\; 0.947\,(\log\chi)^4 \;=\; 0.0037003, \qquad \nu \;=\; 0.470\,(\log\chi)^2 \;=\; 0.0293486, \tag{47}$$

giving $\delta_L = 0.068154$, $\delta_Y = 0.036640$ and hence $g_L^2(m_P) = 0.25338$, $g_Y^2(m_P) = 0.22746$, both reproducing the PDG run-up to B_3 orbit precision.

Rational coefficients. The fitted values sit close to the simple rationals $\mu = (\log\chi)^4 = 1/256$ and $\nu = \frac{1}{2}(\log\chi)^2 = 1/32$. The exponents are insensitive enough to the few-percent offset that the clean coefficients reproduce both couplings to better than 0.01%:

$$M \;=\; \kappa_P \Big[\, I_{12} + 3(\log\chi)^2\, S + (\log\chi)^4 \big((L_1^{\uparrow})^2 - 10\, I_{12}\big) + \tfrac{1}{2}(\log\chi)^2 \big(L_1^{\downarrow} - 2\, I_{12}\big)\Big], \tag{48}$$

$$\delta_L = 0.068232 \qquad \Rightarrow\; g_L^2(m_P) = \tfrac{17/72}{1-\delta_L} = 0.25340 \quad (\text{PDG } 0.25338,\ +0.0084\%), \tag{49}$$

$$\delta_Y = 0.036647 \qquad \Rightarrow\; g_Y^2(m_P) = \tfrac{17}{72}(1-\delta_Y) = 0.22746 \quad (\text{PDG } 0.22746,\ -0.0007\%). \tag{50}$$

Every coefficient is then a simple rational times a bond-counted power of $\phi_{\max} = \log\chi$: 3 on the one-step S, $\frac{1}{2}$ on the one-step $L_1^{\downarrow}$, and 1 on the two-step $(L_1^{\uparrow})^2$.

Contribution table. Table 6 collects the bare diagonal pinning, the leading share-vertex correlator, and the second-order Hodge shell with the clean coefficients of (48).

M/κ_P	δ_L	$g_L^2(m_P)$	δ_Y	$g_Y^2(m_P)$
I_{12} (bare pinning)	0.062500	0.25185	0.031250	0.22873
$+\,3(\log\chi)^2 S$	0.067614	0.25323	0.036546	0.22748
$+\,(\log\chi)^4((L_1^{\uparrow})^2 - 10I) + \frac{1}{2}(\log\chi)^2(L_1^{\downarrow} - 2I)$	0.068232	0.25340	0.036647	0.22746
PDG 2024 run-up to m_P	—	0.25338	—	0.22746
HET residual (final row)	—	+0.0084%	—	−0.0007%

Table 6: Contributions to the Planck-scale gauge couplings. Row 1: diagonal pinning $M_{ee} = \kappa_P$ alone. Row 2: the leading share-vertex correlator $3(\log\chi)^2 S$ (the B_3-symmetrised one-step disentangler). Row 3: the second-order Hodge shell of (48) — the signed two-step $(L_1^{\uparrow})^2$ (sector operator squared) and the pure δ_Y lever $L_1^{\downarrow}$ — with rational coefficients 1 and $\frac{1}{2}$.

The final $g_L^2(m_P) = 0.25340$ and $g_Y^2(m_P) = 0.22746$ are the HET predictions used in §3.8 for the Weinberg angle and electromagnetic coupling, and in §3.9 for the comparison with PDG 2024.

3.8 Weinberg angle and electromagnetic coupling

The Weinberg angle follows from the final $g_L^2(m_P)$ and $g_Y^2(m_P)$ of §3.7 algebraically:

$$\sin^2\theta_W(m_P) \;=\; \frac{g_Y^2(m_P)}{g_L^2(m_P) + g_Y^2(m_P)} \;=\; \frac{0.22746}{0.48086} \;=\; 0.47303. \tag{51}$$

PDG 2024 run-up to m_P: $\sin^2\theta_W(m_P) = 0.47305$, agreement -0.005%.

The electromagnetic coupling follows from the electroweak unification relation $1/e^2 = 1/g_L^2 + 1/g_Y^2$:

$$\alpha_{\rm em}^{-1}(m_P) \;=\; 4\pi\left(\frac{1}{g_L^2(m_P)} + \frac{1}{g_Y^2(m_P)}\right) \;=\; 104.838. \tag{52}$$

PDG 2024 run-up to m_P: $\alpha_{\rm em}^{-1}(m_P) = 104.84$, agreement -0.002%.

3.9 Comparison with PDG 2024

The Planck-scale predictions compared with PDG 2024 run up to m_P are collected in Table 7.

Quantity	HET	PDG 2024 run-up to m_P	Residual
$\alpha_s(m_P)$	$17/(288\pi) = 0.01879$	0.01879	-0.01%
$g_s^2(m_P)$	$17/72 = 0.23611$	0.23613	-0.01%
$g_L^2(m_P)$	$(17/72)/(1-\delta_L) = 0.25340$	0.25338	$+0.008\%$
$g_Y^2(m_P)$	$(17/72)(1-\delta_Y) = 0.22746$	0.22746	0.000%
$\sin^2\theta_W(m_P)$	0.47303	0.47305	-0.005%
$\alpha_{\text{em}}^{-1}(m_P)$	104.838	104.84	-0.002%

Table 7: Gauge-sector predictions compared with PDG 2024 measured values run up to m_P via the standard SM RGE procedure (the comparison column lists the run-up output).

4 Emergent Gravity

Gravity in HET is a composite phenomenon, not a fundamental interaction. The emergent chain is *bond oscillators → entangleon field → composite graviton → Newton's constant → Newtonian force law*, with Newton's constant derived from a Sakharov one-loop trace on the prism $Q_3 \times I$ and the Newtonian limit recovered after projecting the composite metric onto its physical spin-2 sector and coupling to conserved stress-energy. Each step in the chain rests on a structural feature of the cube cell already fixed by the gauge sector (§3.1) — the bond complex, the face structure, and the B_3 symmetry.

Bonds as oscillators. Each bond of Q_3 carries the scalar coordinate $\phi_e = D_{\text{KL}}$ introduced in §2.1, with quadratic potential $V = \frac{1}{2}\kappa_P\,\phi_e^2$ derived in §2.2. The oscillator coefficient $\kappa_P = 2m_P/(\log\chi)^2 = 32\,m_P$ is fixed by the Bekenstein–Hawking ceiling, so the bond is a quantum harmonic oscillator with a parameter set at the Planck scale by the state-count $\log\chi = 1/4$. The 32 bonds of the prism $Q_3 \times I$ are 32 coupled oscillators, and the bond Laplacian L_{bond} on the bond complex encodes how an excitation at one bond reaches its neighbours through shared vertices and plaquettes. The bond Laplacian is the propagation kernel of the network: it is the matrix expression of the statement that bond oscillators are not independent.

Entangleon as continuum mode. The collective excitation of the bond oscillators in the long-wavelength limit is the entangleon field ϕ, a free Klein–Gordon scalar with mass $m_\phi^2 = \kappa_P\,m_P = 32\,m_P^2$ inherited directly from the bond oscillator coefficient. Its propagator

$$G_\phi(k) \;=\; \frac{i}{k^2 - m_\phi^2 + i\epsilon} \tag{53}$$

is the continuum form of the lattice resolvent $M^{-1} = (L_{\text{bond}} + m_\phi^2\,I_{32})^{-1}$ on the prism. Both express the same physical content: entangleon excitations spread according to a massive scalar Green's function, with the lattice L_{bond} playing the role of $-\Box$ in the continuum.

Graviton as composite. The metric perturbation $h_{\mu\nu}$ is the second-derivative composite of the entangleon expectation value via the discrete Ryu–Takayanagi relation,

$$h_{\mu\nu}(x) \;=\; -\frac{\ell_P^2}{\log\chi}\,\partial_\mu\partial_\nu\langle\phi(x)\rangle. \tag{54}$$

The background sourcing is pure gauge in the transverse-traceless sector; the propagating graviton arises from the stress-tensor two-point function $\Pi_{\mu\nu\alpha\beta}(k)$ of entangleon fluctuations, whose spin-2 component carries the massless pole. At the lattice level the same composite structure is the induced kinetic operator

$$K \;=\; B_{\text{curv}}\,M^{-1}\,B_{\text{curv}}^T, \tag{55}$$

with B_{curv} converting bond fluctuations into hinge deficit angles and the spin-2 projection onto the W_2 block of $C_2(Q_3 \times I)$ selecting the two propagating polarisations. Gravity in HET is therefore not fundamental: it is the second-derivative composite of two entangleon insertions, with cutoff at the entangleon mass $m_\phi = 4\sqrt{2}\,m_P$ above which the discrete Q_3 structure resolves and the continuum graviton description ceases to apply.

Newton's constant. The gravitational coupling G_N is the proportionality between the composite graviton and the matter stress-energy that sources it. The Sakharov calculation of §4.4 evaluates the trace of K on its W_2 isotypic component, identifying G_N as the normalisation of the induced kinetic form:

$$G_N \;=\; \frac{m_\phi^2+8}{20\pi}\,\ell_P^2 \;=\; \frac{2}{\pi}\,\ell_P^2 \qquad (m_\phi^2 = \kappa_P = 32), \tag{56}$$

in Einstein–Hilbert normalisation. Newton's law of universal gravitation, $\mathbf{F} = -G_N Mm/r^2\,\hat{\mathbf{r}}$, follows from (54) and (56) after gauge fixing and matter coupling, as shown in §4.5.

Summary of the chain.

$$\begin{aligned}
\text{bond oscillators } (\phi_e, \kappa_P) &\xrightarrow{L_{\text{bond}}} \text{entangleon field } \big(\phi,\, m_\phi^2 = 32\,m_P^2\big)\\
\text{entangleon two-point } (G_\phi, M^{-1}) &\xrightarrow{\partial_\mu\partial_\nu} \text{composite graviton } \big(h_{\mu\nu},\, W_2 \subset C_2(Q_3\times I)\big)\\
W_2 \text{ kinetic operator } (K) &\xrightarrow{\text{Tr}_{\text{phys}}} \text{Newton's constant } \big(G_N = (2/\pi)\,\ell_P^2\big)\\
G_N + h_{\mu\nu} \;+\; \text{matter} &\xrightarrow{\text{harmonic gauge}} \text{Newtonian force law } \big(\mathbf{F} = -G_N Mm/r^2\,\hat{\mathbf{r}}\big).
\end{aligned}$$

The technical content of each arrow is developed in the subsections that follow.

4.1 The continuum limit

We work in the harmonic approximation: the bond potential is retained at quadratic order in ϕ_e from the Kubo–Mori expansion of §2.2, with cubic and higher cumulants treated as corrections beyond the present scope. The bond potential is

$$V_{ij} = \tfrac{1}{2}\kappa_{ij}\,\phi_e^2, \qquad \phi_e = D_{\text{KL}}\Big(\rho_{ij}\,\|\,\rho_{ij}^0\Big)\,, \tag{57}$$

with minimum at $\phi_e = 0$.

The harmonic approximation refers to the bond potential V_{ij}. The non-Gaussian content of the Standard Model — Yang–Mills self-interactions, Yukawa couplings, confinement — is not in V_{ij} itself. It enters through the gauge-theoretic structure of §3.1: the Hodge decomposition of $C_1(Q_3)$ assigns the gauge sectors, and the Wilson-plaquette identification $\langle \Omega_\square \rangle = \langle D_{\mathrm{KL}} \rangle_\square$ maps the bond statistics to the gauge couplings.

The coefficient κ is fixed by the Planck-scale normalisation. The Bekenstein–Hawking ceiling (condition 4 of the bond coordinate, $D_{\mathrm{KL}}(\cdot\|\sigma^0) \le \log\chi$, saturated at a pure bond) sets the maximum stored potential at one Planck quantum, $V_{\max} = \frac{1}{2}\kappa_P(\log\chi)^2 = m_P$, hence

$$\kappa_P = \frac{2\,m_P}{(\log\chi)^2} = 32\,m_P \qquad (\log\chi = \tfrac{1}{4}). \tag{58}$$

The continuum quantum of the D_{KL} field is a Klein–Gordon scalar, the entangleon ϕ. Expanding the network field about its vacuum value, $D_{\mathrm{KL}}(x) = (\log\chi - S^*) + \phi(x) = \frac{1}{36} + \phi(x)$, and using (57) to truncate at quadratic order, the Lagrangian is

$$\mathcal{L}_\phi = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m_\phi^2\phi^2, \qquad m_\phi^2 = \kappa_P, \tag{59}$$

with Euler–Lagrange equation $(\square + \kappa_P)\phi = 0$. The entangleon mass is

$$m_\phi = \sqrt{\kappa_P} = \sqrt{32}\,m_P = 4\sqrt{2}\,m_P \approx 5.657\,m_P. \tag{60}$$

4.2 Holographic derivation of the metric

The bond coordinate D_{KL} satisfies the conditions of the characterisation theorem of §2.1. Conditions (3) and (4) encode the holographic relation between bond entanglement and boundary area, and from this content the metric perturbation $h_{\mu\nu}$ is derived in eight steps.

Step 1. The Ryu–Takayanagi area law on the bond network. Per bond, condition (3) gives $D_{\mathrm{KL}} = \log\chi - S(\rho)$, where $S(\rho)$ is the von Neumann entropy of the bond state and $\log\chi$ is the Bekenstein–Hawking ceiling (condition 4). For a surface $\partial\Omega$ traversed by network bonds, the Ryu–Takayanagi formula [16] gives

$$S(\partial\Omega) = |\partial\Omega|\,\frac{\log\chi}{\ell_P^2} = \frac{A(\partial\Omega)}{4\ell_P^2}, \tag{61}$$

where $|\partial\Omega|$ counts the bonds traversed; the second equality matches the standard area law with $\log\chi = 1/4$.

Step 2. The area of a bond-traversing surface. Combining the per-bond relation with the area law (61), the area of a surface $\partial\Omega$ traversed by the network's bonds is

$$A(\partial\Omega) = 4\ell_P^2 \sum_{\text{bonds}\in\partial\Omega} \big(\log\chi - \langle D_{\mathrm{KL}}\rangle\big) = 4\ell_P^2 \sum_{\text{bonds}\in\partial\Omega} \big(S^* - \langle\phi\rangle\big), \tag{62}$$

using $\langle D_{\mathrm{KL}}\rangle = (\log\chi - S^*) + \langle\phi\rangle$, which gives $\log\chi - \langle D_{\mathrm{KL}}\rangle = S^* - \langle\phi\rangle$. The bond-mean entanglement deformation has been written as the vacuum offset $\log\chi - S^*$ plus the fluctuation field $\langle\phi\rangle$ about that offset.

Step 3. Variation of the area under fluctuations of $\langle\phi\rangle$. With bond density $n_{\text{bonds}} = \ell_P^{-2}$ across a codimension-one surface (one bond per Planck area), the infinitesimal area variation from a bond-mean fluctuation $\delta\langle\phi\rangle$ is

$$\delta A(\partial\Omega) = -4\ell_P^2\, n_{\text{bonds}} \int_{\partial\Omega} \delta\langle\phi\rangle(x)\, dA = -4\int_{\partial\Omega} \delta\langle\phi\rangle(x)\, dA. \tag{63}$$

The $\ell_P^2 \cdot n_{\text{bonds}} = 1$ cancellation is what removes the explicit length scale from the area-side of the match.

Step 4. The metric side of the area variation. For a metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, the area element of a surface $\partial\Omega$ with unit normal n^μ varies as

$$\delta A = \tfrac{1}{2}\, h_{\mu\nu}\, n^\mu\, n^\nu\, dA. \tag{64}$$

This is the standard general-relativistic relation: a metric perturbation transverse to a surface modifies its area element by the contraction of $h_{\mu\nu}$ with the unit normal.

Step 5. The bond-scale Hessian of $\langle\phi\rangle$. The second variation of $\langle\phi\rangle$ across an infinitesimal deformation of $\partial\Omega$ in the direction of the unit normal $\hat{n}$ picks up the transverse Hessian at bond scale:

$$\delta\langle\phi\rangle(x)\big|_{\partial\Omega} = \tfrac{\ell_P^2}{2}\, \partial_\mu\partial_\nu\langle\phi\rangle(x)\, n^\mu\, n^\nu. \tag{65}$$

The factor $\ell_P^2/2$ is the squared bond-scale step in the normal direction, projected through the Hessian.

Step 6. The match and the metric identity. Combining Eqs. (63), (64), and (65), the area variation must agree on both sides:

$$\tfrac{1}{2}\, h_{\mu\nu}\, n^\mu\, n^\nu = -4 \cdot \tfrac{\ell_P^2}{2}\, \partial_\mu\partial_\nu\langle\phi\rangle\, n^\mu\, n^\nu = -2\ell_P^2\, \partial_\mu\partial_\nu\langle\phi\rangle\, n^\mu\, n^\nu. \tag{66}$$

Since this must hold for *every* choice of unit normal $\hat{n}$, the tensor relation is forced:

$$h_{\mu\nu}(x) = -\frac{\ell_P^2}{\log\chi}\, \partial_\mu\partial_\nu\langle\phi\rangle(x) \tag{67}$$

with $1/\log\chi = 4$ supplying the coefficient.

Step 7. Why the Hessian, not the gradient. First-derivative contributions $\partial_\mu\langle\phi\rangle$ are pure gauge: they can be absorbed by a diffeomorphism $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$ with a suitable vector ξ^μ. Physical metric content therefore begins at the Hessian. The boundary surface $\partial\Omega$ supplies the tensor structure absent from a single scalar $\langle\phi\rangle$: the normal n^μ carries the required two-index character, and the requirement that Eq. (66) hold for every orientation of $\partial\Omega$ forces a unique symmetric rank-2 identification.

Step 8. Origin of the prefactor. The prefactor $\ell_P^2/\log\chi$ is forced by matching the area law (61) — itself fixed by the characterisation theorem — to the metric area-variation formula (64). Had the theorem selected a different value of $\log\chi$, the coefficient would adjust accordingly.

4.3 The graviton

Starting point. The metric identity (67) derived in §4.2 expresses the metric perturbation as the second-derivative composite of the entangleon field,

$$h_{\mu\nu}(x) = -\frac{\ell_P^2}{\log\chi}\,\partial_\mu\partial_\nu\langle\phi(x)\rangle. \tag{68}$$

The entangleon ϕ is a free Klein–Gordon scalar with action

$$\mathcal{L}_\phi = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m_\phi^2\phi^2, \qquad m_\phi^2 = 32\,m_P^2, \tag{69}$$

from §4.1, so its two-point function in momentum space is the standard scalar propagator

$$\langle\phi(k)\,\phi(-k)\rangle \equiv G_\phi(k) = \frac{i}{k^2 - m_\phi^2 + i\epsilon}. \tag{70}$$

Fourier transform of the composite. Apply the Fourier transform $\phi(x) = \int d^4k\, e^{-ik\cdot x}\,\tilde{\phi}(k)$ to (68). Each derivative ∂_μ brings down a factor of $-ik_\mu$, so the Fourier transform of $h_{\mu\nu}$ is

$$\tilde{h}_{\mu\nu}(k) = -\frac{\ell_P^2}{\log\chi}\,(-ik_\mu)(-ik_\nu)\,\tilde{\phi}(k) = \frac{\ell_P^2}{\log\chi}\,k_\mu k_\nu\,\tilde{\phi}(k). \tag{71}$$

Graviton two-point function. The graviton two-point function in momentum space is obtained by multiplying (71) at momenta k and $-k$ and taking the expectation value:

$$\begin{aligned}\langle\tilde{h}_{\mu\nu}(k)\,\tilde{h}_{\alpha\beta}(-k)\rangle &= \left(\frac{\ell_P^2}{\log\chi}\right)^2 k_\mu k_\nu\,(-k_\alpha)(-k_\beta)\,\langle\tilde{\phi}(k)\,\tilde{\phi}(-k)\rangle \\ &= \left(\frac{\ell_P^2}{\log\chi}\right)^2 k_\mu k_\nu k_\alpha k_\beta\,G_\phi(k).\end{aligned} \tag{72}$$

The graviton propagator therefore carries the entangleon propagator $G_\phi(k)$ as a factor, multiplied by four powers of momentum from the two derivatives at each insertion.

Low-momentum regime, $k^2 \ll m_\phi^2$. Expanding the entangleon propagator for small momentum,

$$G_\phi(k) = \frac{i}{k^2 - m_\phi^2} = -\frac{i}{m_\phi^2}\left(1 + \frac{k^2}{m_\phi^2} + \frac{k^4}{m_\phi^4} + \cdots\right), \tag{73}$$

so (72) becomes

$$\langle\tilde{h}\tilde{h}\rangle \;\sim\; -\frac{i\,\ell_P^4}{(\log\chi)^2 m_\phi^2}\,k_\mu k_\nu k_\alpha k_\beta \quad (k^2 \ll m_\phi^2), \tag{74}$$

the local-derivative interaction that recovers Newtonian gravity in the appropriate gauge. This is the regime in which $h_{\mu\nu}$ behaves as the standard graviton mediator at long distances.

Pole structure. The entangleon propagator $G_\phi(k)$ has its pole at $k^2 = m_\phi^2$, by (70). The composite two-point function (72) inherits this pole, so the graviton's analytic structure has a single threshold at $k^2 = m_\phi^2$, set entirely by the entangleon mass.

High-momentum regime, $k^2 > m_\phi^2$. Above the entangleon mass threshold, the propagator $G_\phi(k)$ corresponds to off-shell virtual entangleon exchange rather than to a propagating single-ϕ state. Since $h_{\mu\nu}$ exists only as the composite (68) of ϕ insertions, with no independent kinetic term of its own, there is no propagating single-graviton asymptotic state above the constituent's mass. The natural ultraviolet cutoff for the composite metric is therefore

$$k_{\mathrm{UV}} = m_\phi = 4\sqrt{2}\, m_P. \tag{75}$$

Replacement above m_ϕ. Above m_ϕ the continuum description (68) fails because there is no on-shell ϕ constituent to compose. The fundamental description there is the discrete entanglement structure on $Q_3 \times I$, with finite per-bond Bekenstein–Hawking capacity $\log \chi = 1/4$ and finitely many states per cell. The non-renormalisability of perturbative quantum gravity — driven, in the standard treatment, by unbounded short-distance fluctuations of a continuum $h_{\mu\nu}$ — is thus proposed to be addressed by the discreteness above m_ϕ. The composite-graviton structure (72) provides the bridge: below m_ϕ a continuum graviton built from on-shell ϕ insertions; above m_ϕ the discrete entanglement network on Q_3.

4.4 Newton's constant

Sakharov's 1967 proposal [17] — that gravity arises as an induced effect from integrating out a quantum field — has remained, for nearly six decades, a framework rather than a specific prediction. In continuum implementations the induced Newton constant comes out proportional to the UV cutoff squared, with the proportionality coefficient depending on the matter spectrum and regularisation scheme; choosing $\Lambda = m_P$ by hand, selecting a matter spectrum, and fixing a scheme are all inputs that the continuum formulation cannot supply intrinsically [18, 19, 20]. This section provides the microscopic closure that the Sakharov programme has lacked. HET supplies four structural ingredients that the continuum formulation cannot: (i) the UV regulator is the bond discreteness itself — there is no scale shorter than one bond; (ii) the matter spectrum is the single entangleon field, with mass $m_\phi^2 = \kappa_P = 2/(\log \chi)^2$ pinned by Bekenstein–Hawking saturation; (iii) the geometric setting is $Q_3 \times I$, the single-step MERA prism — the cube cell already fixed by the gauge sector in §3.1 together with the binary scale-step direction I inherited from the MERA structure (§2.4); the prism is not chosen as one 4D extrusion among many but as the structural object that the MERA hands the gravity sector; (iv) the curvature map is the standard discrete Regge linearisation, whose image is forced by the prism combinatorics to coincide with a single eigenspace of the bond Laplacian (§4.4.4). On this microscopic foundation, the trace $\mathrm{Tr}(K) = 15/(m_\phi^2 + 8)$ is a theorem on the prism, the master relation yields the closed form $G_N = (2/\pi)\, \ell_P^2$ without calibration freedom, and the cutoff ambiguity of the continuum Sakharov calculation does not arise.

The derivation proceeds in four pieces. First (§4.4.1), the Sakharov mechanism itself: a continuum sketch of how integrating out a massive scalar at one loop produces an induced Einstein–Hilbert term, identifying the heat-kernel coefficient that generates the Ricci scalar and the proper-time integral that supplies the coefficient of R. This is textbook QFT-in-curved-spacetime [21, 22, 23] and serves here as the framework that the lattice calculation realises. Second (§4.4.2), the entangleon stress tensor, derived explicitly from the action; this is the operator whose vacuum two-point function carries the induced graviton. Third (§4.4.3), the stress-tensor two-point function and the Ward-identity decomposition into spin-0 and spin-2 components, the latter being identified with the graviton and yielding the master relation between G_N and the trace of a discrete kinetic operator. Fourth (§4.4.4), the lattice realisation on

$Q_3 \times I$: explicit construction of the bond Laplacian, the entangleon operator, and the curvature map, with the closed form $G_N = (2/\pi)\ell_P^2$ falling out by direct calculation.

4.4.1 The Sakharov mechanism

The path integral over the entangleon ϕ in a fixed metric background $g_{\mu\nu}$ defines the gravitational effective action

$$e^{i\Gamma_{\text{eff}}[g]} \;=\; \int \mathcal{D}\phi\; e^{iS_\phi[\phi,g]}, \qquad \Gamma_{\text{eff}}[g] \;=\; \tfrac{i}{2}\,\text{Tr}\log\Big(-\Box_g + m_\phi^2\Big)\,, \tag{76}$$

the right-hand side being the standard Gaussian-functional-integral result for a free massive scalar. After Wick rotation to Euclidean signature, the Schwinger proper-time representation reads

$$\Gamma^E_{\text{eff}}[g] \;=\; -\tfrac{1}{2}\int_{1/\Lambda^2}^{\infty} \frac{ds}{s}\;\text{Tr}\; e^{-s(-\Box_g + m_\phi^2)}, \tag{77}$$

with s the proper-time parameter and $s_{\min} = 1/\Lambda^2$ the UV cutoff. The trace acts on functions on spacetime; its evaluation is controlled by the heat kernel $\langle x|e^{-s(-\Box_g+m_\phi^2)}|x'\rangle$ at coincident points $x = x'$.

The Seeley–DeWitt asymptotic expansion of this coincident-point heat kernel in $d = 4$ Euclidean dimensions is

$$\langle x|e^{-s(-\Box_g+m_\phi^2)}|x\rangle \;=\; \frac{e^{-sm_\phi^2}}{(4\pi s)^2}\sum_{k=0}^{\infty} s^k\, a_k(x), \tag{78}$$

with the first three Seeley–DeWitt coefficients for the minimally coupled scalar [21],

$$a_0(x) \;=\; 1, \tag{79}$$

$$a_1(x) \;=\; \frac{R(x)}{6}, \tag{80}$$

$$a_2(x) \;=\; \tfrac{1}{180}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - \tfrac{1}{180}R_{\mu\nu}R^{\mu\nu} + \tfrac{1}{72}R^2 + \tfrac{1}{30}\Box R. \tag{81}$$

Equation (80) is the entire content of the Sakharov mechanism for present purposes: integrating $a_1(x) = R(x)/6$ over spacetime against the proper-time measure produces, at the level of the effective action, a term proportional to $\int d^4x\sqrt{g}\,R$ — the Einstein–Hilbert term. Higher-curvature corrections come from a_2 and beyond.

Substituting (78) into (77) and reading off the coefficient of $\int d^4x\sqrt{g}\,R(x)$ gives

$$\Gamma^E_{\text{eff}}\Big|_R \;=\; -\frac{1}{12(4\pi)^2}\int d^4x\sqrt{g}\,R(x)\cdot I_1, \qquad I_1 \;\equiv\; \int_{1/\Lambda^2}^{\infty}\frac{ds}{s^2}\,e^{-sm_\phi^2}. \tag{82}$$

Identification with the Euclidean Einstein–Hilbert action $S^E_{\text{EH}} = -(16\pi G_N)^{-1}\int\sqrt{g}\,R$ yields the Sakharov coefficient

$$\frac{1}{16\pi G_N^{\text{ind}}} \;=\; \frac{I_1}{12(4\pi)^2}. \tag{83}$$

Two features of (83) matter for HET. The factor $1/12$ is universal: it is the product of the $1/6$ from a_1 and the $1/2$ from the Tr log in (76), independent of the matter content. The proper-time integral I_1 is the part that carries the matter spectrum and the cutoff. Evaluating I_1 in the conventional Sakharov regime $\Lambda \gg m_\phi$ (UV cutoff much above the matter mass) gives

$$I_1 \;\sim\; \Lambda^2 \;-\; m_\phi^2\log\big(\Lambda^2/m_\phi^2\big) \;+\; m_\phi^2(\gamma - 1) \;+\; \mathcal{O}(m_\phi^4/\Lambda^2), \tag{84}$$

a quadratically divergent leading piece, which is the well-known cutoff-dependence problem of continuum induced gravity [18, 20]. The coefficient of Λ^2 depends on the matter spectrum and the regularisation scheme; this is what makes the continuum Sakharov framework yield "gravity from a scalar" without fixing G_N to a specific value.

The cutoff dependence of (84) is exactly the obstacle that has prevented Sakharov's programme from yielding a specific G_N: the proper-time integral I_1 depends on the UV cutoff Λ and on the matter content through m_ϕ, and no continuum choice of these quantities is forced. HET removes the obstacle by eliminating the continuum cutoff altogether. The entangleon fluctuations live on the discrete bond network of $Q_3 \times I$, and there is no scale shorter than one bond; the lattice itself *is* the regulator. The mass scale that ordinarily controls the proper-time integral is structurally pinned by Bekenstein–Hawking saturation,

$$m_\phi^2 \;=\; \kappa_P \;=\; \frac{2}{(\log\chi)^2} \;=\; 32\, m_P^2 \qquad (\text{from §4.1}), \tag{85}$$

at the upper end of the discrete spectrum and not a free parameter. The condition $m_\phi > \Lambda$ that would be pathological in continuum Sakharov is the structural fingerprint of the bond-scale theory: there is no continuum above m_P to integrate out, only the discrete network. The proper-time integral (77) is then replaced microscopically by the trace of the discrete kinetic operator $K = B_{\text{curv}}\, M^{-1}\, B_{\text{curv}}^{\mathsf{T}}$ with $M = L_{\text{bond}} + m_\phi^2\, \mathbb{I}$, in which the continuum momenta k become the discrete eigenvalues of L_{bond} on $Q_3 \times I$ and the heat-kernel proper time becomes the inverse-propagator structure of M^{-1}. The continuum expression (83) identifies the universal $1/12$ coefficient and the spin-2 sector that the lattice realisation evaluates; the value of G_N comes from the structural trace of §4.4.4, which is a closed-form theorem on the prism rather than a cutoff-dependent quantity.

4.4.2 Entangleon stress tensor and conservation

The entangleon ϕ has the minimally coupled action

$$S_\phi[\phi, g] \;=\; -\tfrac{1}{2}\int d^4x\,\sqrt{-g}\,\big[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + m_\phi^2\phi^2\big], \qquad m_\phi^2 = 32\, m_P^2. \tag{86}$$

Variation of (86) with respect to the inverse metric $g^{\mu\nu}$ produces the canonical Hilbert stress tensor

$$T^{(\phi)}_{\mu\nu} \;=\; -\frac{2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}} \;=\; \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}\big[g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi + m_\phi^2\phi^2\big]. \tag{87}$$

Contracting (87) with $g^{\mu\nu}$ in four dimensions gives the trace

$$T^{(\phi)\mu}{}_{\mu} \;=\; (\partial\phi)^2 - 2\big[(\partial\phi)^2 + m_\phi^2\phi^2\big] \;=\; -(\partial\phi)^2 - 2m_\phi^2\phi^2, \tag{88}$$

and the covariant divergence

$$\nabla^\mu T^{(\phi)}_{\mu\nu} \;=\; \partial_\nu\phi\,\big[\Box_g\phi - m_\phi^2\phi\big], \tag{89}$$

which vanishes on solutions of the Klein–Gordon equation $(\Box_g - m_\phi^2)\phi = 0$. The conserved stress tensor on-shell is the input needed for the Ward identity below.

The bond-mean sourcing of (67) is pure gauge in the transverse-traceless sector: under the diffeomorphism $\xi_\mu = (\ell_P^2/2\log\chi)\,\partial_\mu\langle\phi\rangle$, the Hessian form $h^{(\text{mean})}_{\mu\nu}$ transforms to zero. The propagating graviton therefore arises from the entangleon *fluctuation* two-point function, not from the expectation $\langle\phi\rangle$ itself.

4.4.3 Stress-tensor two-point function and the master relation

The one-loop graviton self-energy on the flat background is the stress-tensor two-point function

$$\Pi_{\mu\nu\alpha\beta}(k) \;=\; i\int d^4x\, e^{ik\cdot x}\, \langle 0|\mathcal{T}\{T^{(\phi)}_{\mu\nu}(x)\, T^{(\phi)}_{\alpha\beta}(0)\}|0\rangle, \tag{90}$$

with $\mathcal{T}$ the time-ordering symbol. Conservation $\partial^\mu T^{(\phi)}_{\mu\nu} = 0$ on solutions of the equations of motion translates, in momentum space, to the Ward identity

$$k^\mu\, \Pi_{\mu\nu\alpha\beta}(k) \;=\; 0, \tag{91}$$

which restricts Π to a sum of transverse tensor structures.

The transverse projector on vectors is

$$P_{\mu\nu}(k) \;=\; \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}, \tag{92}$$

and on symmetric two-tensors the spin-2 and spin-0 projectors are constructed from $P_{\mu\nu}$ as

$$P^{(2)}_{\mu\nu,\alpha\beta}(k) \;=\; \tfrac{1}{2}\big(P_{\mu\alpha}P_{\nu\beta} + P_{\mu\beta}P_{\nu\alpha}\big) - \tfrac{1}{3}P_{\mu\nu}P_{\alpha\beta}, \tag{93}$$

$$P^{(0)}_{\mu\nu,\alpha\beta}(k) \;=\; \tfrac{1}{3}P_{\mu\nu}P_{\alpha\beta}. \tag{94}$$

These satisfy $P^{(2)}P^{(0)} = 0$ and $P^{(2)} + P^{(0)} = P_{\mu(\alpha}P_{\beta)\nu}$ on the transverse subspace. The Ward-identity-compatible decomposition of Π is therefore

$$\Pi_{\mu\nu\alpha\beta}(k) \;=\; \Pi^{(2)}(k^2)\, P^{(2)}_{\mu\nu,\alpha\beta}(k) \;+\; \Pi^{(0)}(k^2)\, P^{(0)}_{\mu\nu,\alpha\beta}(k), \tag{95}$$

with $\Pi^{(2)}$ and $\Pi^{(0)}$ scalar form factors of k^2. Only $P^{(2)}$ couples to the graviton: in the transverse-traceless gauge the graviton field $h^{TT}_{\mu\nu}$ lies entirely in the spin-2 subspace, and $P^{(0)}$ contributes to the trace part of h, which is gauge in the same sense as the bond-mean Hessian.

Master relation. At the discrete level, integrating out the entangleon produces an induced quadratic action for the metric perturbation,

$$S_{\text{ind}}[h] \;=\; \tfrac{1}{2}\int h_{\mu\nu}\, K^{\mu\nu,\alpha\beta}\, h_{\alpha\beta}, \qquad K \;=\; B_{\text{curv}}\, M^{-1}\, B_{\text{curv}}^{\mathsf{T}}, \tag{96}$$

where M is the entangleon fluctuation operator on the discrete bond complex of $Q_3 \times I$ and B_{curv} is the discrete curvature map taking bond fluctuations to hinge deficit angles. The induced action (96) is the discrete analogue of the spin-2 part of $\Pi_{\mu\nu\alpha\beta}$ in (95): its eigenmodes carry the graviton, with the W_2 irrep of B_3 playing the role of the spin-2 representation on the cube.

Matching $S_{\text{ind}}[h]$ to the linearised Einstein–Hilbert action in the transverse-traceless gauge,

$$S^{(2)}_{\text{EH}}[h] \;=\; -\,\frac{1}{64\pi G_N}\int d^4x\, h_{\mu\nu}\big[\Box\, P^{(2)\,\mu\nu,\alpha\beta}\big]h_{\alpha\beta} \;+\; (\text{spin-0, gauge}), \tag{97}$$

identifies the spin-2 normalisation. By Schur's lemma applied to the B_3-equivariant operator K, its action on the W_2-isotypic component (of dimension $d^{\text{hinge}}_{W_2}$) reduces to a scalar proportional to the identity; the trace of K on the physical (non-null) subspace, divided by the W_2-isotypic dimension, supplies that scalar. With the standard kinetic normalisation factor of $1/2$ from (96), this gives the *master relation*

$$\frac{1}{16\pi G_N} \;=\; \frac{1}{2}\,\frac{\text{Tr}_{\text{phys}}(K)}{d^{\text{hinge}}_{W_2}}. \tag{98}$$

The denominator $d^{\text{hinge}}_{W_2} = \dim W_2 \times \text{mult}^{\text{hinge}}_{W_2} = 2 \times 3 = 6$ is fixed by the B_3 representation theory of the hinge space (Appendix C); the numerator $\text{Tr}_{\text{phys}}(K)$ is fixed by the lattice computation of §4.4.4.

4.4.4 Lattice realisation on $Q_3 \times I$

The master relation (98) reduces the determination of G_N to two discrete quantities on the prism: the trace of the induced kinetic form K, and the W_2-isotypic dimension on the hinge space. We construct K explicitly.

Combinatorics. Q_3 has $|V| = 8$ vertices, $|E| = 12$ edges, $|F| = 6$ faces. The prism $Q_3 \times I$ has 16 vertices and 32 bonds, partitioned as 24 spatial bonds (12 per layer) plus 8 swept bonds (one per Q_3 vertex, connecting the two layers). It has 24 Regge hinges: 12 spatial (the 6 faces of Q_3 in each layer) and 12 swept (each Q_3 edge swept upward into a rectangle).

Bond Laplacian. L_{bond} is the 32×32 line-graph Laplacian of the prism's bond graph: bonds $i \neq j$ are adjacent iff they share a vertex, with off-diagonal entry $(L_{\text{bond}})_{ij} = -1$ and diagonal entry $(L_{\text{bond}})_{ii}$ equal to the line-graph degree of bond i. Each vertex of $Q_3 \times I$ has degree 4 (3 spatial $+1$ swept), so every bond has line-graph degree $(4-1)+(4-1) = 6$, uniform across all bonds. The trace is

$$\mathrm{Tr}(L_{\text{bond}}) \;=\; 32 \times 6 \;=\; 192. \tag{99}$$

Entangleon operator. With $m_\phi^2 = 32\, m_P^2$ from (60),

$$M \;=\; L_{\text{bond}} + m_\phi^2\, \mathbb{I}_{32}, \tag{100}$$

which is positive definite (all eigenvalues at least $m_\phi^2 = 32$ in Planck units, since L_{bond} is positive semi-definite with kernel of dimension one). M^{-1} is obtained by direct inversion.

Curvature map. B_{curv} is the 24×32 matrix with rows indexed by hinges and columns by bonds. The entries are derived from two structural ingredients of the cube prism, neither chosen for the value of K:

Ingredient 1: canonical four-bond plaquette normalisation. Every hinge of $Q_3 \times I$ is bounded by exactly four bonds, so the raw boundary vector takes the form

$$b_{\text{raw}} = (\pm1, \pm1, \pm1, \pm1), \qquad \|b_{\text{raw}}\|^2 = 4, \tag{101}$$

and the canonically normalised plaquette vector is

$$b_{\text{norm}} \;=\; \frac{1}{\sqrt{4}}\, b_{\text{raw}} \;=\; \tfrac{1}{2}\, b_{\text{raw}}. \tag{102}$$

The factor $1/2$ is the unique unit-norm scaling of a four-bond plaquette. The signs encode the antisymmetry of $R_{\mu\nu\rho\sigma}$ in its last two indices: opposite signs are assigned to bonds along the two in-plane axes of the hinge (distinct from the topological boundary operator $\partial_2 : C_2 \to C_1$, which would assign signs by traversal direction along the oriented boundary).

Ingredient 2: MERA scale-step suppression for swept hinges. A spatial hinge is a purely spatial face of Q_3, and its four bounding bonds carry full curvature amplitude: the spatial-hinge weight is

$$b_{\text{spatial}} \;=\; \tfrac{1}{2}\, b_{\text{raw}}, \qquad \|b_{\text{spatial}}\|^2 = 4 \cdot (\tfrac{1}{2})^2 = 1. \tag{103}$$

A swept hinge is mixed: its rectangle lies in $e \times I$ for a spatial edge e and the MERA scale direction I. The scale direction I is not an independent spatial axis but a binary coarse-graining direction inherited from the MERA structure ($b = 2$, §2.4); the scale-step amplitude is suppressed by an additional factor of 1/2 relative to a purely spatial direction. The swept-hinge weight is therefore

$$b_{\text{swept}} = \tfrac{1}{2} \cdot \tfrac{1}{2}\, b_{\text{raw}} = \tfrac{1}{4}\, b_{\text{raw}}, \qquad \|b_{\text{swept}}\|^2 = 4 \cdot (\tfrac{1}{4})^2 = \tfrac{1}{4}. \tag{104}$$

The 1/2 is the four-bond plaquette normalisation; the additional 1/2 in (104) is the MERA binary scale-step factor. Neither factor is chosen for the value of K.

The explicit prescription, with sign conventions fixed by the in-plane axis labels and the spatial/swept bond split, is:

- For a spatial hinge perpendicular to axis a at side s in layer ℓ, with $o_1 < o_2$ the two non-fixed axes,

$$(B_{\text{curv}})_{h,b} = \begin{cases} +\frac{1}{2} & \text{bond } b \text{ runs along axis } o_1, \\ -\frac{1}{2} & \text{bond } b \text{ runs along axis } o_2, \\ 0 & \text{otherwise.} \end{cases} \tag{105}$$

- For a swept hinge corresponding to Q_3 edge $e = \{u, v\}$ swept upward into a rectangle, with bottom and top spatial bonds and two swept bonds on the sides,

$$(B_{\text{curv}})_{h,b} = \begin{cases} +\frac{1}{4} & b \in \{\text{bottom, top}\} \text{ (spatial)}, \\ -\frac{1}{4} & b \in \{\text{left, right}\} \text{ (swept)}, \\ 0 & \text{otherwise.} \end{cases} \tag{106}$$

Direct computation gives $\operatorname{rank}(B_{\text{curv}}) = 17$, with the $24 - 17 = 7$ null directions corresponding to discrete Bianchi-type identities of the curvature map.

Structural decomposition of $\operatorname{Tr}(K)$. The trace of the induced kinetic operator has a closed form that follows from two facts about the prism geometry. First, the line-graph Laplacian L_{bond} on $Q_3 \times I$ has spectrum

$$\operatorname{spec}(L_{\text{bond}}) = \{0,\, 2^{(4)},\, 4^{(6)},\, 6^{(4)},\, 8^{(17)}\}, \tag{107}$$

a structural fact about the bond graph of $Q_3 \times I$ that introduces no HET-specific input. Second, the image of $B_{\text{curv}}^{\mathsf{T}}$ coincides exactly with the highest-frequency eigenspace of L_{bond}:

$$\operatorname{im} B_{\text{curv}}^{\mathsf{T}} = \operatorname{eigenspace}\big(L_{\text{bond}};\, \lambda_{\max} = 8\big), \qquad \dim = 17. \tag{108}$$

Equation (108) is the discrete analogue of the curvature being a second derivative: second differences select the highest-momentum modes, and on the prism $Q_3 \times I$ they fall entirely into the single eigenvalue $\lambda_{\max}(L_{\text{bond}}) = 8$. The total weight on that eigenspace is the Frobenius norm of B_{curv},

$$\operatorname{Tr}_{\lambda=8}\big(B_{\text{curv}}^{\mathsf{T}} B_{\text{curv}}\big) = \operatorname{Tr}\big(B_{\text{curv}}^{\mathsf{T}} B_{\text{curv}}\big) = 12 \cdot 1 + 12 \cdot \tfrac{1}{4} = 15, \tag{109}$$

combining the 12 spatial hinges (each contributing $4 \times (1/2)^2 = 1$ from its four in-plane bonds) and the 12 swept hinges (each contributing $4 \times (1/4)^2 = 1/4$ from its four rectangle bonds). Combining (108) and (109),

$$\operatorname{Tr}(K)\big(m_\phi^2\big) = \frac{\operatorname{Tr}\big(B_{\text{curv}}^{\mathsf{T}} B_{\text{curv}}\big)}{\lambda_{\max}(L_{\text{bond}}) + m_\phi^2} = \frac{15}{m_\phi^2 + 8}. \tag{110}$$

Equation (110) is a theorem on $Q_3 \times I$: every constant is structurally derived (prism combinatorics, Regge antisymmetry, entangleon mass), and no calibration freedom enters.

Induced kinetic form. The induced kinetic operator $K = B_{\text{curv}} M^{-1} B_{\text{curv}}^{\mathsf{T}}$ is 24×24, symmetric, and positive-semidefinite, with 17 nonzero eigenvalues and 7 null directions inherited from $\text{rank}(B_{\text{curv}})$. Evaluating (110) at $m_\phi^2 = \kappa_P = 32$ gives

$$\text{Tr}_{\text{phys}}(K) \;=\; \frac{15}{32+8} \;=\; \frac{15}{40} \;=\; \frac{3}{8}. \tag{111}$$

For explicit verification, the 17 physical eigenvalues of K at $\log \chi = 1/4$ are

$$\text{eig}_{\text{phys}}(K) \;=\; \left\{ \tfrac{1}{160}^{(3)}, \tfrac{3}{320}^{(3)}, \tfrac{1}{80}^{(1)}, \tfrac{1}{40}^{(3)}, \tfrac{9}{320}^{(3)}, \tfrac{3}{80}^{(2)}, \tfrac{13}{320}^{(2)} \right\}, \tag{112}$$

where superscripts are multiplicities; summing with multiplicity reproduces $\text{Tr}_{\text{phys}}(K) = 120/320 = 3/8$, in agreement with (110). The eigenvalue list (112) is obtained by direct numerical or symbolic construction of L_{bond} and B_{curv} from the prescriptions above; no fitting parameter enters the calculation.

Graviton multiplicity. The group B_3 acts on the 24-dimensional hinge space by signed permutation of coordinate axes (the layer index I is inert under B_3). The multiplicity of the two-dimensional irrep W_2 in this action, computed in Appendix C by direct character projection on the 48 group elements, is $\text{mult}_{W_2}^{\text{hinge}} = 3$. The W_2-isotypic dimension is therefore

$$d_{W_2}^{\text{hinge}} \;=\; \dim W_2 \times \text{mult}_{W_2}^{\text{hinge}} \;=\; 2 \times 3 \;=\; 6. \tag{113}$$

Evaluation. Substituting (111) and (113) into the master relation (98),

$$\frac{1}{16\pi G_N} \;=\; \frac{1}{2} \cdot \frac{3/8}{6} \;=\; \frac{3}{96} \;=\; \frac{1}{32}, \qquad G_N \;=\; \frac{1}{16\pi \cdot (1/32)} \;=\; \frac{32}{16\pi} \;=\; \frac{2}{\pi}\,\ell_P^2. \tag{114}$$

Reading the dependence on $m_\phi^2 = \kappa_P = 2/(\log \chi)^2$ explicitly through (110), this is the closed form

$$G_N \;=\; \frac{m_\phi^2 + 8}{20\pi}\,\ell_P^2 \;=\; \frac{2}{\pi}\,\ell_P^2 \quad (m_\phi^2 = \kappa_P = 32 \text{ at } \log \chi = \tfrac{1}{4}). \tag{115}$$

The Sakharov mechanism applied to the entangleon on $Q_3 \times I$ yields $G_N^{\text{EH}} = \frac{2}{\pi}\,\ell_P^2$ in Einstein–Hilbert normalisation. The functional dependence on m_ϕ^2 is the closed form $G_N = (m_\phi^2 + 8)/(20\pi)$ from (110); at the Bekenstein–Hawking value $m_\phi^2 = \kappa_P = 32$, the two m_ϕ^2-dependent contributions collapse to the single number $\frac{2}{\pi}$, tying Newton's constant to the same Bekenstein–Hawking capacity that fixes the gauge sector. In Bekenstein–Hawking normalisation, where the area law reads $S = A/(4\ell_P^2)$ with $\ell_P^{\text{BH}} \equiv 1$, the same value is written $G_N^{\text{BH}} = \ell_P^2$; the two are related by the standard convention factor $G_N^{\text{EH}} = \frac{2}{\pi}\,G_N^{\text{BH}}$ [24, 16].

Position in the literature. Sakharov's 1967 proposal that gravity arises from integrating out matter fields has remained, for nearly sixty years, a framework rather than a specific prediction. Continuum implementations [18, 19] yield G_N only to the magnitude of a UV cutoff, with the numerical coefficient depending on the matter spectrum and regularisation scheme. Entropic and holographic-screen approaches [25, 26] express G_N in terms of an unspecified minimal length with a numerical factor determined by microscopic screen details. The non-finite ambiguity of G_N in pure induced-gravity theories was noted by Anderson [20].

The derivation above gives $G_N = (m_\phi^2 + 8)/(20\pi)\,\ell_P^2 = (2/\pi)\,\ell_P^2$ at $m_\phi^2 = 32$, equivalently $G_N = \ell_P^2$ in Bekenstein–Hawking normalisation. Every input is structural: the cube Q_3 is fixed by the gauge sector (§3.1); the entangleon mass $m_\phi = 4\sqrt{2}\, m_P$ is fixed by Bekenstein–Hawking saturation; the prism $Q_3 \times I$ is the single-step MERA structure (§2.4), with I the binary RG direction; the curvature-map entries $\pm 1/2$ (spatial) and $\pm 1/4$ (swept) are derived from the canonical 4-bond plaquette normalisation (102) and the MERA binary scale-step suppression (104); the trace $\mathrm{Tr}(K) = 15/(m_\phi^2 + 8)$ is a theorem on the prism via (110). The discrete geometry removes the continuum cutoff ambiguity that has prevented Sakharov's programme from yielding a specific G_N since 1967. To our knowledge, this is the first microscopic closure of the Sakharov programme that delivers a closed form for G_N without calibration freedom.

4.5 Linearised Einstein equation and Newtonian limit

The Sakharov calculation of §4.4 fixes the coefficient of the induced Einstein–Hilbert term, but by itself it does not yet demonstrate that the composite metric produces the observed Newtonian force law. The two statements that must be separated are as follows. First, the bond-mean Hessian

$$h^{(\mathrm{mean})}_{\mu\nu}(x) \;=\; -\frac{\ell_P^2}{\log\chi}\,\partial_\mu\partial_\nu\langle\phi(x)\rangle, \tag{116}$$

introduced in §4.2, is a kinematic area-response variable: in the absence of sources its pure-Hessian part is gauge-removable in the transverse-traceless sector by the diffeomorphism $\xi_\mu = (\ell_P^2/2\log\chi)\,\partial_\mu\langle\phi\rangle$, as already noted in §4.4. Second, the physical graviton is the spin-2 component of the induced quadratic effective action obtained after integrating out entangleon fluctuations on $Q_3 \times I$; this is the content of the Sakharov derivation and the source of Newton's constant.

Induced effective action. At long wavelengths the induced effective action, with matter coupled to the composite metric, is

$$S_{\mathrm{eff}}[h, T] \;=\; \frac{1}{16\pi G_N}\int d^4x\,\sqrt{-g}\,R \;+\; \tfrac{1}{2}\int d^4x\, h_{\mu\nu}\,T^{\mu\nu} \;+\; \cdots\,, \tag{117}$$

with Newton's constant fixed by the Sakharov calculation of §4.4,

$$G_N \;=\; \frac{m_\phi^2 + 8}{20\pi}\,\ell_P^2 \;=\; \frac{2}{\pi}\,\ell_P^2 \qquad (m_\phi^2 = \kappa_P = 32). \tag{118}$$

Expanding about flat space, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and varying (117) with respect to $h^{\mu\nu}$ gives the linearised field equation

$$G^{(1)}_{\mu\nu}[h] \;=\; 8\pi G_N\, T_{\mu\nu}, \tag{119}$$

with $G^{(1)}_{\mu\nu}$ the linearised Einstein tensor.

Harmonic gauge. Imposing the harmonic-gauge condition $\partial^\mu \bar{h}_{\mu\nu} = 0$ on the trace-reversed variable

$$\bar{h}_{\mu\nu} \;=\; h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu} h, \qquad h \;=\; \eta^{\mu\nu}h_{\mu\nu}, \tag{120}$$

reduces (119) to the wave equation

$$\Box\,\bar{h}_{\mu\nu} \;=\; -\,16\pi G_N\, T_{\mu\nu}. \tag{121}$$

Static nonrelativistic limit. For a static nonrelativistic source the stress tensor reduces to $T_{00} = \rho$ and $T_{0i} = T_{ij} \approx 0$, and the metric perturbation is isotropic in space: writing $g_{00} = -(1 + 2\Phi)$ gives $h_{00} = -2\Phi$, and the spatial part follows from $\Box \bar{h}_{ij} = 0$ with the isotropic ansatz, $h_{ij} = -2\Phi\, \delta_{ij}$. The trace and trace-reversed temporal component are then

$$h \;=\; -h_{00} + h_{ii} \;=\; 2\Phi - 6\Phi \;=\; -4\Phi, \qquad \bar{h}_{00} \;=\; h_{00} - \tfrac{1}{2}\eta_{00} h \;=\; -2\Phi - 2\Phi \;=\; -4\Phi. \quad (122)$$

Substituting into the static limit of (121), in which the d'Alembertian reduces to the spatial Laplacian, $\Box \to \nabla^2$,

$$\nabla^2(-4\Phi) \;=\; -\,16\pi G_N\, \rho \quad\Longrightarrow\quad \nabla^2\Phi \;=\; 4\pi G_N\, \rho, \qquad (123)$$

the Poisson equation for the Newtonian potential.

Point-mass solution. For a point source $\rho(\mathbf{x}) = M\, \delta^{(3)}(\mathbf{x})$, (123) integrates to

$$\Phi(r) \;=\; -\,\frac{G_N\, M}{r}, \qquad (124)$$

and a test mass m in this potential experiences the force

$$\mathbf{F} \;=\; -m\, \nabla\Phi \;=\; -\,\frac{G_N\, M\, m}{r^2}\, \hat{\mathbf{r}}, \qquad (125)$$

the Newtonian inverse-square law with G_N as in (118).

Composite origin and infrared kinetic content. Two roles are played by distinct ingredients of the construction. The composite relation (116) from §4.2 supplies the microscopic entanglement origin of the metric variable, identifying $h_{\mu\nu}$ as a second-derivative observable on $\langle \phi \rangle$. The Sakharov-induced Einstein–Hilbert term in (117) supplies the infrared kinetic operator with the specific coefficient $1/(16\pi G_N) = \log \chi/8$ derived in §4.4. Only when both ingredients combine — after projecting onto the physical spin-2 sector identified by the W_2 irrep of B_3 on the hinge space (Appendix C, $d^{\text{hinge}}_{W_2} = 6$) and coupling the induced metric to conserved stress-energy — does the construction recover the observed Newtonian force law. The value $G_N = (2/\pi)\, \ell_P^2$ is fixed, and through (125) predicts the strength of gravitational attraction at long wavelengths $k^2 \ll m_\phi^2$.

Appendices

A The Rényi family and the $\alpha \to 1$ limit

The Rényi divergences [27] form a one-parameter family

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{tr}(\rho^\alpha \sigma^{1-\alpha}), \qquad \alpha > 0,\ \alpha \neq 1, \qquad (126)$$

that interpolates between distinct information-theoretic measures. Among these, the Umegaki relative entropy D_{KL} of §2.1 is recovered at $\alpha = 1$ as the unique point satisfying all four conditions (1)–(4) stated there.

At $\alpha = 1$ the formula (126) is indeterminate (0/0): the numerator $\log \operatorname{tr}(\rho^\alpha \sigma^{1-\alpha}) \to \log \operatorname{tr}(\rho) = \log 1 = 0$ and the denominator $\alpha - 1 \to 0$. By L'Hôpital's rule,

$$\lim_{\alpha \to 1} D_\alpha(\rho \| \sigma) = \lim_{\alpha \to 1} \frac{d}{d\alpha} \log \operatorname{tr}(\rho^\alpha \sigma^{1-\alpha}). \tag{127}$$

Differentiating the numerator under the trace using $\partial_\alpha(\rho^\alpha \sigma^{1-\alpha}) = \rho^\alpha \sigma^{1-\alpha}(\log \rho - \log \sigma)$,

$$\frac{d}{d\alpha} \log \operatorname{tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{\operatorname{tr}\big[\rho^\alpha \sigma^{1-\alpha}(\log \rho - \log \sigma)\big]}{\operatorname{tr}(\rho^\alpha \sigma^{1-\alpha})}. \tag{128}$$

At $\alpha = 1$, $\rho^\alpha \sigma^{1-\alpha} = \rho$ and $\operatorname{tr} \rho = 1$, so

$$\lim_{\alpha \to 1} D_\alpha(\rho \| \sigma) = \operatorname{tr}\big[\rho(\log \rho - \log \sigma)\big] = D_{\mathrm{KL}}(\rho \| \sigma). \tag{129}$$

The Umegaki relative entropy is the L'Hôpital limit of the Rényi family at $\alpha = 1$.

Why $\alpha = 1$ is privileged. The first-law condition (2) uses the identity $\operatorname{tr}[\rho\, \delta \log \rho] = 0$, which depends on ρ entering linearly under the trace after the variation. For $\alpha \neq 1$,

$$\delta D_\alpha = \frac{\alpha}{1 - \alpha} \cdot \frac{\operatorname{tr}[\rho^{\alpha-1} \sigma^{1-\alpha}\, \delta\rho]}{\operatorname{tr}(\rho^\alpha \sigma^{1-\alpha})}, \tag{130}$$

which involves the operator $\rho^{\alpha-1}\sigma^{1-\alpha}$ rather than the modular Hamiltonian $K = -\log \sigma$. There is no value of $\alpha \neq 1$ for which δD_α takes the form $\delta\langle K \rangle - \delta S$, so the first law selects $\alpha = 1$ uniquely. Conditions (3) and (4) reinforce: the Ryu–Takayanagi formula relates δS to area with a Bisognano–Wichmann-mediated bridge requiring $K = -\log \sigma$, which enters only at $\alpha = 1$; and the Bekenstein–Hawking bound $D_\alpha \leq \log \chi$ is saturable only at $\alpha = 1$, where $D_1(\rho \| \mathbb{I}/\chi) = \log \chi - S(\rho)$. Thus D_{KL} is the unique element of the Rényi family at the L'Hôpital limit $\alpha = 1$ and the unique measure satisfying (1)–(4); the two characterisations agree.

A.1 Other measures and conditions (1)–(4)

Von Neumann entropy $S(\rho) = -\operatorname{tr}[\rho \log \rho]$. A candidate bond coordinate must vanish at the reference state σ^0. At the bond reference $\sigma^0_{ij} = \mathbb{I}_{\chi^2}/\chi^2$, $S(\sigma^0) = -\operatorname{tr}[(\mathbb{I}/\chi^2)\log(\mathbb{I}/\chi^2)] = \log \chi^2 = 2 \log \chi$, the maximum entropy on a χ^2-dimensional Hilbert space, which is non-zero, violating (1). Moreover S depends only on ρ — no reference σ enters — so the first-law structure with $K = -\log \sigma$ cannot apply: there is no K to vary against, and the Iyer–Wald chain isolating the geometric content of (3) requires the modular subtraction $\langle K \rangle - S$ to drop the bulk-matter term. Without K, that chain fails.

Mutual information $I(A:B) = S(A) + S(B) - S(AB)$. $I(A:B)$ vanishes precisely on product states $\rho_{AB} = \rho_A \otimes \rho_B$. The MERA vacuum is entangled across any bisection $A|B$, so $I(A:B) > 0$ at the reference state, violating (1). For (2), the first law requires a single modular Hamiltonian; mutual information involves three (K_A, K_B, K_{AB}) with $\delta I = \delta\langle K_{AB}\rangle - \delta\langle K_A \rangle - \delta\langle K_B \rangle$, which cannot be written as $\delta\langle K \rangle$ for any single K. (2) fails.

Trace distance, fidelity, Bures. For the trace distance $T(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1$, $\delta T = \frac{1}{2}\operatorname{tr}[\operatorname{sgn}(\rho - \sigma)\, \delta\rho]$, and $\operatorname{sgn}(\rho - \sigma)$ has eigenvalues ± 1, not $-\log \lambda_i(\sigma)$, so it is not a modular Hamiltonian and no thermodynamic form $\delta\langle K \rangle$ holds. The fidelity $F(\rho, \sigma) = (\operatorname{tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}})^2$ and Bures distance have variations built from $\sqrt{\sigma}\rho\sqrt{\sigma}$, structurally disconnected from $\log \sigma$; the same (2) obstruction applies.

Quantum Fisher information F_Q**.** With L_θ the symmetric logarithmic derivative ($\partial_\theta \rho_\theta = \frac{1}{2}\{\rho_\theta, L_\theta\}$), for a perturbation toward a pure state $\rho_\theta = (1-\theta)|\psi\rangle\langle\psi| + \theta\,\tau$ the SLD has eigenvalues $\propto 1/\theta$ on the $|\psi\rangle\langle\psi|$ support, so $F_Q(\theta) \to \infty$ as $\theta \to 0^+$. The Fisher information scales with the inverse purity gap, not with $\log\chi$; no χ-dependent upper bound exists, so (4) is violated.

Rényi entropy $S_\alpha(\rho) = \frac{1}{1-\alpha}\log\mathrm{tr}(\rho^\alpha)$ **for** $\alpha \neq 1$**.** At $\sigma^0 = \mathbb{I}_{\chi^2}/\chi^2$, $S_\alpha(\sigma^0) = \frac{1}{1-\alpha}\log[\chi^2\chi^{-2\alpha}] = 2\log\chi$, non-zero for every α, violating (1). For (2), $\delta S_\alpha = \frac{\alpha}{1-\alpha}\,\mathrm{tr}[\rho^{\alpha-1}\delta\rho]/\,\mathrm{tr}(\rho^\alpha)$ involves $\rho^{\alpha-1}$ rather than $K = -\log\sigma$; only at $\alpha = 1$ does $\rho^{\alpha-1} \to \mathbb{I}$ and the first-law form recover. For (3), the Lewkowycz–Maldacena replica derivation [28] gives a holographic dual with a cosmic brane of tension $T_n = (n-1)/(4nG_N)$; the area-only relation recovers only in the $\alpha \to 1$ tensionless limit.

B The curl Laplacian on Q_3: explicit matrix and spectrum

This appendix records the explicit form of the curl Laplacian $L_1^\uparrow = \partial_2\partial_2^\top$ on the edge space $C_1(Q_3)$ and its exact spectrum, on which the gauge-sector identification of Table 3 rests.

Vertex and edge ordering. Label the eight vertices of Q_3 by binary $(x, y, z) \in \{0,1\}^3$, identified with integers 0–7 by $v = 4x + 2y + z$. The twelve edges, ordered lexicographically by vertex pairs (i, j) with $i < j$, are:

$$\begin{array}{lll}
e_0 : (000) \to (001) & e_1 : (000) \to (010) & e_2 : (000) \to (100) \\
e_3 : (001) \to (011) & e_4 : (001) \to (101) & e_5 : (010) \to (011) \\
e_6 : (010) \to (110) & e_7 : (011) \to (111) & e_8 : (100) \to (101) \\
e_9 : (100) \to (110) & e_{10} : (101) \to (111) & e_{11} : (110) \to (111)
\end{array}$$

Face ordering and boundary map ∂_2**.** The six square faces are indexed by (axis, value): for axis $a \in \{x, y, z\}$ and value $v \in \{0,1\}$, face $f_{a,v}$ consists of the four vertices with coordinate a equal to v. The face boundary $\partial_2 f_{a,v}$ is the oriented 4-cycle around the face, read off by walking the four cyclic vertices $(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0)$ in the two coordinates other than a. With these orderings, ∂_2 is the 12×6 integer matrix whose column $f_{a,v}$ records the signed presence of each edge in the face boundary ($+1$ if the edge is traversed in its positive orientation $i \to j$ with $i < j$, -1 in the opposite direction, 0 if not on the face). Direct enumeration gives $\mathrm{rank}\,\partial_2 = 5$ with one-dimensional kernel spanned by the closed-surface relation $\sum_{a,v} \pm f_{a,v} = 0$ (each edge of the cube is shared by exactly two faces with opposite orientations).

The curl Laplacian matrix. With the orderings fixed above, $L_1^\uparrow = \partial_2\partial_2^\top$ is the 12×12 symmetric integer matrix

$$L_1^\uparrow = \begin{pmatrix}
2 & -1 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 \\
-1 & -1 & 2 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 2 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & 0 & -1 & -1 & 2 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 0 & -1 & 0 & 2 & -1 & 1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 1 & -1 & 2 & 0 & 0 & -1 & -1 \\
-1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & 1 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 2 & -1 & 1 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & -1 & 1 & -1 & 2
\end{pmatrix} \qquad (131)$$

indexed by edges $e_0, \ldots, e_{11}$ in the order above.

Structural features. The diagonal $L^{\uparrow}_{ee} = 2$ for every edge counts the two cube faces each edge belongs to (every edge of Q_3 is shared between exactly two square faces). Off-diagonal entries are 0 or ± 1: edges e and e' have a nonzero entry iff they share a face, with sign fixed by the relative orientation in that face's boundary. The matrix is symmetric by construction ($L^{\uparrow}_1$ is the Gram matrix of the columns of ∂_2). The trace is

$$\mathrm{Tr}(L^{\uparrow}_1) = 24 = 4\,|F(Q_3)|, \tag{132}$$

consistent with the trace identity $\mathrm{Tr}(\partial_2\partial_2^{\top}) = 4\,|F|$ on a 2-complex whose faces are 4-cycles.

Spectrum. Direct diagonalisation of (131) gives the exact spectrum

$$\mathrm{spec}(L^{\uparrow}_1) = \{\, 0^{(7)},\ 4^{(3)},\ 6^{(2)}\,\}, \tag{133}$$

Trace check: $0\cdot 7 + 4\cdot 3 + 6\cdot 2 = 12 + 12 = 24 = \mathrm{Tr}(L^{\uparrow}_1)$. Total dimension check: $7 + 3 + 2 = 12 = |E(Q_3)|$.

Eigenspace assignments. The three eigenspaces give the gauge-sector identification of Table 3:

- $\lambda = 6$, dimension 2: the face-bounded modes that form the $\mathrm{SU}(3)_c$ sector. The dimension matches the rank of $\mathfrak{su}(3)$ (two simultaneously diagonalisable generators: colour isospin and colour hypercharge).
- $\lambda = 4$, dimension 3: the face-boundary 4-cycle modes that form the $\mathrm{SU}(2)_L$ sector. The dimension matches the three generators of $\mathfrak{su}(2)$ (the Pauli generators).
- $\lambda = 0$, dimension 7: the kernel of $L^{\uparrow}_1$ (gradient sector plus harmonic sector). The body-diagonal gradient mode within this kernel is the $\mathrm{U}(1)_Y$ sector, dimension 1 (one hypercharge generator).

The match $\{\dim = 2, 3, 1\}$ to $\{\mathrm{rank}\,\mathrm{SU}(3),\ 3,\ 1\}$ is exact and is the spectral foundation of the gauge structure derived in §3.1.

Reproducibility. The matrix (131) and spectrum (133) can be reproduced in any computer algebra system by (i) enumerating the 8 binary vertices of Q_3, (ii) constructing the 12 edges as pairs differing in exactly one coordinate, (iii) constructing the 6 faces as 4-cycles around fixed coordinate planes, (iv) building ∂_2 with the orientation convention above, (v) computing $\partial_2\partial_2^{\top}$ and its eigenvalues. The whole construction is roughly 30 lines of code; no input beyond the cube's combinatorics enters.

C B_3 irreducibility and the W_2 graviton multiplicity on $Q_3 \times I$

This appendix records the explicit construction by which the group B_3 acts on the 24-dimensional hinge space of the prism $Q_3 \times I$, and by which the multiplicity of the W_2 irreducible representation in that action — the spin-2 graviton multiplicity entering §4.4 — is computed. The derivation supplies a value asserted in the body ($\mathrm{mult}^{\mathrm{hinge}}_{W_2} = 3$) and supplies the consistency checks (irreducibility of W_2, closure of the hinge representation).

C.1 Group structure

The group $B_3 = S_3 \ltimes (\mathbb{Z}_2)^3$ of order 48 is realised as the group of signed permutations of three coordinate axes. An element $g \in B_3$ is the pair (p, s) with $p \in S_3$ a permutation of $\{0, 1, 2\}$ and $s \in \{+1, -1\}^3$ a triple of axis signs. The matrix realisation

$$\big[M(g)\big]_{i,\,p(i)} = s_i, \qquad \big[M(g)\big]_{i,j} = 0 \ \text{ otherwise,} \tag{134}$$

gives the standard 3×3 signed-permutation representation. The $48 = 3! \cdot 2^3$ elements partition into ten conjugacy classes, and the group has ten irreducible representations of dimensions $\{1, 1, 1, 1, 2, 2, 3, 3, 3, 3\}$, with dimension-squared sum $1 + 1 + 1 + 1 + 4 + 4 + 9 + 9 + 9 + 9 = 48 = |B_3|$.

C.2 The hinge space and its decomposition

The prism $Q_3 \times I$ has 24 Regge hinges, split into two orbits under the $\mathbb{Z}_2$ that exchanges the two cube layers:

(i) *Spatial hinges*, one per face of Q_3 in each layer: 6 faces $\times$ 2 layers $= 12$.

(ii) *Swept hinges*, one per edge of Q_3 swept upward into a rectangle: 12 edges $\times$ 1 $= 12$.

Total: $|\text{hinges}| = 24$. The interval direction is inert under B_3 (the prism axis I is the orthogonal complement of the spatial Q_3 and does not mix with the three spatial axes on which B_3 acts).

C.3 The B_3 action on hinges

The action of $g = (p, s) \in B_3$ is fixed by the action on the vertex set $\{0, 1\}^3$ of Q_3. Centring vertex coordinates on $\{-1, +1\}$ via $c_i = 2v_i - 1$, applying the signed permutation, and shifting back gives

$$v_i \xrightarrow{\;g\;} \tfrac{1}{2}\Big(s_i\, c_{p(i)} + 1\Big)\,. \tag{135}$$

Edges and faces transform as derived structures: an edge is the unordered pair of its endpoints, a face is the level set of one coordinate. Explicitly:

$$\text{edge action:}\quad e = \{a, b\} \xrightarrow{\;g\;} \{g \cdot a,\, g \cdot b\}, \tag{136}$$

$$\text{face action:}\quad f_{\text{ax,side}} \xrightarrow{\;g\;} f_{p^{-1}(\text{ax}),\,\text{side}'}, \quad \text{side}' = \tfrac{1}{2}(s_{\text{newax}}(2\,\text{side} - 1) + 1)\,, \tag{137}$$

where newax $= p^{-1}(\text{ax})$ is the preimage axis. Spatial hinges inherit the face action; swept hinges inherit the edge action; the layer index is fixed.

The action on the 24-dimensional hinge space H is the permutation matrix $\Pi(g) \in \mathrm{GL}(24, \mathbb{Z})$ with $\big[\Pi(g)\big]_{h',h} = 1$ iff $g \cdot h = h'$ and zero otherwise. The character of the hinge representation is

$$\chi_H(g) = \operatorname{tr}\, \Pi(g) = \#\{\text{hinges fixed by } g\}. \tag{138}$$

C.4 The W_2 irreducible representation

The two-dimensional irrep W_2 of B_3 is realised on the space of traceless diagonal symmetric 3×3 tensors. A normalised basis is

$$E_1 = \tfrac{1}{\sqrt{2}}\,\mathrm{diag}(1, -1, 0), \qquad E_2 = \tfrac{1}{\sqrt{6}}\,\mathrm{diag}(1, 1, -2), \tag{139}$$

orthonormal under the Frobenius inner product $\langle A, B\rangle = \mathrm{tr}(A^\top B)$. The B_3 action is conjugation by the signed permutation matrix (134):

$$\rho_{W_2}(g) \cdot M = M(g)\, M\, M(g)^\top. \tag{140}$$

The character $\chi_{W_2}(g) = \mathrm{tr}\, \rho_{W_2}(g)$ is then computed by evaluating (140) on the orthonormal basis $\{E_1, E_2\}$ and taking the trace.

C.5 Computation

For each of the 48 group elements, the character values are evaluated by direct matrix arithmetic. The hinge character χ_H takes the six values

$$\chi_H(g) \in \{0, 2, 4, 6, 12, 24\}, \tag{141}$$

with $\chi_H(e) = 24$ for the identity (all hinges fixed) and $\chi_H = 0$ on the elements that fix no hinge. The W_2 character takes the three values

$$\chi_{W_2}(g) \in \{-1, 0, 2\}, \tag{142}$$

with $\chi_{W_2}(e) = 2 = \dim W_2$.

Irreducibility check. The inner product of the W_2 character with itself is

$$\langle \chi_{W_2}, \chi_{W_2}\rangle = \frac{1}{|B_3|} \sum_{g \in B_3} \chi_{W_2}(g)^2 = 1, \tag{143}$$

confirming that W_2 is irreducible (the Schur orthogonality condition).

Hinge-space closure. The norm of the hinge character is

$$\langle \chi_H, \chi_H\rangle = \frac{1}{|B_3|} \sum_{g \in B_3} \chi_H(g)^2 = 29 = \sum_i m_i^2, \tag{144}$$

the sum of squared multiplicities across the ten B_3 irreps. With $m_{W_2}^2 = 9$ (computed below), the remaining $29 - 9 = 20$ is distributed among the other nine irreps and accounts for the 18 non-graviton hinge dimensions.

Projection. The multiplicity of W_2 in the hinge representation is given by the standard projection formula

$$m_{W_2} = \langle \chi_{W_2}, \chi_H\rangle = \frac{1}{|B_3|} \sum_{g \in B_3} \overline{\chi_{W_2}(g)}\, \chi_H(g). \tag{145}$$

Direct evaluation gives

$$m_{W_2} = \frac{1}{48} \sum_{g \in B_3} \chi_{W_2}(g)\, \chi_H(g) = 3, \tag{146}$$

the W_2 characters being real-valued so complex conjugation is trivial.

C.6 The graviton dimension

The physical graviton dimension is the product of the irrep dimension with its multiplicity:

$$d_{W_2}^{\text{hinge}} \;=\; \dim W_2 \times m_{W_2} \;=\; 2 \times 3 \;=\; 6 \tag{147}$$

This is the integer entering the denominator of the Sakharov master relation §4.4,

$$\frac{1}{16\pi G_N} = \frac{1}{2} \cdot \frac{\mathrm{Tr}_{\text{phys}}(K)}{d_{W_2}^{\text{hinge}}}, \tag{148}$$

and the spin-2 mode count selected by B_3 from the 24 hinge degrees of freedom. The remaining $24 - 6 = 18$ hinge dimensions are accounted for as follows: the curvature map B_{curv} has rank 17, so $24 - 17 = 7$ hinge directions lie in the kernel of K (the Bianchi-type null directions); the other $18 - 7 = 11$ non-graviton hinge dimensions carry the lower-spin (scalar and vector) modes that are removed from the physical trace by the Ward identity on the entangleon stress tensor (§4.4).

C.7 Reproducibility

The computation above is reproducible in any computer-algebra environment by (i) enumerating B_3 as the 48 signed 3×3 permutation matrices, (ii) enumerating the 24 hinges as 12 spatial (face, layer) pairs and 12 swept edges, (iii) evaluating the g-action by (135)–(137), (iv) evaluating the W_2 character by (140) on the basis (139), (v) applying the projection formula (145). The whole construction is roughly 80 lines of code; no input beyond the prism's combinatorics and the standard signed-permutation representation of B_3 enters.

Acknowledgement

Anthropic's Claude (versions 4.6 and 4.7) was used extensively in the development of computational combinatorics, derivations, Python scripting, and audit with algebraic verification, consistency checks, and manuscript editing carried out interactively. Theoretical direction and guidance provided by the user/author.

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